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ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

# THE THEORY OF SIGNAL DETECTABILITY

## PART II. APPLICATIONS WITH GAUSSIAN NOISE

ISSUED SEPARATELY:

### PART I. THE GENERAL THEORY

Technical Report No. 13
Electronic Defense Group
Department of Electrical Engineering

EDG

By: W. W. Peterson
    T. G. Birdsall

Approved by H. W. Welch, Jr.

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                                          Project Engineer


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PART II.  APPLICATIONS WITH GAUSSIAN NOISE

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## PART I.  THE GENERAL THEORY (Issued Separately)

## LIST OF ILLUSTRATIONS

# ABSTRACT

## PART I

The several statistical approaches to the problem of
signal detectability which have appeared in the literature are
shown to be essentially equivalent. A general theory based on like-
lihood ratio embraces the criterion approach, for either restricted
false alarm probability or minimum weighted error type optimum, and
the a posteriori probability approach. Receiver reliability is
shown to be a function of the distribution functions of likelihood
ratio. The existence and uniqueness of solutions for the various
approaches is proved under general hypothesis.

## PART II

The full power of the theory of signal detectability can
be applied to detection in Gaussian noise, and several general re-
sults are given. Six special cases are considered, and the
expressions for likelihood ratio are derived. The resulting opti-
mum receivers are evaluated by the distribution functions of the
likelihood ratio. In two of the special cases studied, the uncer-
tainty of the signal ensemble can be varied, throwing some light on
the effect of uncertainty on probability of detection.

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## ACKNOWLEDGMENTS

## THE THEORY OF SIGNAL DETECTABILITY

### Part II. APPLICATIONS WITH GAUSSIAN NOISE

ISSUED SEPARATELY:

### Part I. THE GENERAL THEORY

## 3. INTRODUCTION AND GAUSSIAN NOISE

### 3.1 Introduction

The chief conclusion obtained from the general theory of signal detectability presented in Part I is that a receiver which calculates the likelihood ratio for each receiver input is the optimum receiver. The receiver can be evaluated (e.g., false alarm probability and probability of detection can be found) if the distribution functions for likelihood ratio are known. It is the purpose of Part II to consider a number of different ensembles of signals with Gaussian noise. For each case, a possible receiver design is discussed. The primary emphasis, however, is on obtaining the distribution functions for likelihood ratio, and hence on estimates of receiver performance for the various cases.

The special cases which are presented were chosen from the simplest problems in signal detection which closely represent practical situations. They are listed in Table I along with examples of engineering problems in which they find application.

TABLE I

| Section | Description of Signal Ensemble | Application |
|---------|-------------------------------|-------------|
| 4.2 | Signal Known Exactly[1] | Coherent radar with a target of known range and character |
| 4.3 | Signal Known Except for Phase[1] | Ordinary pulse radar with no integration and with a target of known range and character. |
| 4.4 | Signal a Sample of White Gaussian Noise | Detection of noise-like signals; detection of speech sounds in Gaussian noise. |
| 4.5 | Video Design of a Broad Band Receiver | Detecting a pulse of known starting time (such as a pulse from a radar beacon) with a crystal-video or other type broad band receiver. |
| 4.6 | A Radar Case (A train of pulses with incoherent phase)[2] | Ordinary pulse radar with integration and with a target of known range and character. |
| 4.8 | Signal One of M Orthogonal Signals | Coherent radar where the target is at one of a finite number of non-overlapping positions. |
| 4.9 | Signal One of M Orthogonal Signals Known Except for Phase | Ordinary pulse radar with no integration and with a target which may appear at one of a finite number of non-overlapping positions. |

[1] Our treatment of these two fundamental cases is based upon Woodward and Davies' work, but here they are treated in terms of likelihood ratio, and hence apply to criterion type receivers as well as to a posteriori probability type receivers.

[2] This is essentially the case treated by Middleton in Ref. 7.

In the last two cases the uncertainty in the signal can be varied, and some light is thrown on the relationship between uncertainty and the ability to detect signals.[1] The variety of examples presented should serve to suggest methods for attacking other simple signal detection problems and to give insight into problems too complicated to allow a direct solution.

It should be borne in mind that this report discusses the detection of signals in noise; the problem of obtaining information from signals or about signals, except as to whether or not they are present, is not discussed. Furthermore, in treating the special cases, the noise was assumed to be Gaussian.[2]

The reader will probably find the discussion of likelihood ratio and its distribution easier to follow if he keeps in mind the connection between a criterion type receiver and likelihood ratio. In an optimum criterion type system, the operator will say that a signal is present whenever the likelihood ratio is above a certain level $\beta$. He will say that only noise is present when the likelihood ratio is below $\beta$. For each operating level $\beta$, there is a false alarm probability and a probability of detection. The false alarm probability is the probability that the likelihood ratio $\ell(x)$ will be greater than $\beta$ if no signal is sent; this is by definition the complementary distribution function $F_N(\beta)$. Likewise, the complementary distribution $F_{SN}(\beta)$ is the probability that $\ell(x)$ will be greater than $\beta$ if there is signal plus noise, and hence $F_{SN}(\beta)$ is the probability of detection if a signal is sent.

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[1] The only discussion in the literature on the effect of uncertainty on signal detectability which has come to our attention is in Davies, Ref. 2, where the effect upon signal detectability of not knowing carrier phase is shown quantitatively.

[2] See the footnote on page 4 with reference to the spectrum of the assumed noise.

### 3.2  Gaussian Noise

Throughout Part II, receiver input voltages, which are functions of time, are assumed to be defined for all times t in an observation interval, $0 \leq t \leq T$. They are also assumed to be limited to a band of frequencies of width W. By the sampling theorem,[1] each receiver input can be thought of as a point in a 2WT dimensional space, the coordinates of the point being the value of the function at the "sample points" $t = \frac{1}{2W}$, for $1 \leq i \leq 2WT$. The notation x(t), or simply x, denotes a receiver input, and $x_i$ denotes the $i^{th}$ sample value, or coordinate. The signal as it would appear at the receiver input in the absence of noise is denoted by s(t), or simply s, and the coordinates, or sample values, of s are denoted by $s_i$. The receiver input, which may be due to noise alone or to signal plus noise, is random because of the presence of noise. Therefore, only the probability distribution for the receiver inputs x(t) can be specified. The distribution must be given for the receiver inputs both when there is noise alone and when there is signal plus noise. The probability distributions are described in this report by giving the probability density function $f_{SN}(x)$ and $f_N(x)$ for the receiver inputs x in the 2WT dimensional space.

The noise considered in Part II is always Gaussian noise limited to the bandwidth W, and having a uniform spectrum over the band.[2] This is ordinarily called white Gaussian noise. The probability density function for white Gaussian noise, and hence for the receiver inputs when there is noise alone, is:

$$f_N(x) = \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi N}} \exp\left[ -\frac{x_i^2}{2N} \right] \right\}, \text{ or} \qquad (3.1)$$

---

[1] See Appendix D.

[2] If the noise spectrum is band limited, but not uniform, the noise and signals can be put through a filter which makes the noise uniform, and then the theory can be applied. See H. W. Bode and C. E. Shannon, "A Simplified Derivation of Linear Least Square Smoothing and Prediction Theory," Proc. I.R.E., Vol. 38, p. 417, April 1950.

$$f_N(x) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2N} \sum x_i^2\right]^1 \qquad (3.2a)$$

where n is the dimension of the space, i.e., $2WT$,[2] and N is the noise power.[3]
It can be shown that this ensemble of noise functions has a Gaussian distribution at every time and that its spectrum is uniform.

By the sampling theorem,[4]

$$\sum x_i^2 = 2W \int_0^T \left[x(t)\right]^2 dt \quad . \qquad (3.3)$$

Therefore

$$f_N(x) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{N_0} \int_0^T x(t)^2 dt\right] , \qquad (3.2b)$$

where $N_0 = \frac{N}{W}$ is the noise power per unit bandwidth.[5]

If the signals and their probabilities are known, then the signal plus
noise probability density function, $f_{SN}(x)$, can be found by the convolution
integral, as described in Section 2.[6]

---

[1] Unless otherwise indicated, the limits on the sum are $i = 1$ to $i = n = 2WT$.

[2] If $\frac{1}{2\pi N} \exp\left[-\frac{x_i^2}{2N}\right]$ is called $f_N(x_i)$, then $f_N(x) = \prod_{i=1}^{n} f_N(x_i)$, i.e., the $x_i$ are
independent and each has $f_N(x_i)$ for its probability density function. For a
discussion of "independent," see Cramér, Ref. 14, p. 159.

[3] This assumes the circuit impedance is normalized to one ohm.

[4] See Appendix D.

[5] This form of the expression $f : f_N(x)$, and the corresponding forms of the
equations for $f_{SN}(x)$ and $\mathcal{L}(x)$ were first derived by Woodward. See Woodward
and Davies, Refs. 2 and 3.

[6] See page 13 of Part I.

$$f_{SN}(x) = \int_R f_N(x-s) dP_S(s) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \int_R \exp\left[-\frac{1}{2N} \sum_{i=1}^{n} (x_i - s_i)^2\right] dP_S(s)$$

$$= \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2N} \sum_{i=1}^{n} x_i^2\right] \int_R \exp\left[-\frac{1}{2N} \sum_{i=1}^{n} s_i^2\right] \exp\left[\frac{1}{N} \sum_{i=1}^{n} x_i s_i\right] dP_S(s)$$

<div align="right">(3.4a)</div>

$$f_{SN}(x) = \int_R f_N(x-s) dP_S(s) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \int_R \exp\left[-\frac{1}{N_0} \int_0^T [x(t) - s(t)]^2 dt\right] dP_S(s)$$

<div align="right">(3.4b)</div>

$$= \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{N_0} \int_0^T x^2 dt\right] \int_R \exp\left[-\frac{1}{N_0} \int_0^T s^2 dt\right] \exp\left[\frac{2}{N_0} \int_0^T xs \, dt\right] dP_S(s)$$

The factor $\exp\left[-\frac{1}{N_0} \int_0^T x^2(t) \, dt\right] = \exp\left[-\frac{1}{2N} \sum x_i^2\right]$ can be brought out of

the integral since it does not depend on $s$, the variable of integration. Note
that the integral

$$\int_0^T s^2 dt = \frac{1}{2N} \sum s_i^2 = E(s) \quad [1] \tag{3.5}$$

is the energy of the expected signal, while

$$\int_0^T x(t) s(t) \, dt = \frac{1}{2N} \sum x_i s_i \tag{3.6}$$

is the cross correlation between the expected signal and the receiver input.

[1] See footnote 3 on page 5.

### 3.3  Likelihood Ratio with Gaussian Noise

Likelihood ratio is defined as the ratio of the probability density functions $f_{SN}(x)$ and $f_N(x)$. With white Gaussian noise it is obtained by dividing Eq (3.4) by Eq (3.2).

$$\ell(x) = \int_R \exp\left[-\frac{E(s)}{N_0}\right] \exp\left[\frac{1}{N} \sum_{i=1}^{n} x_i s_i\right] dP_S(s), \quad \text{or} \tag{3.7a}$$

$$\ell(x) = \int_R \exp\left[-\frac{E(s)}{N_0}\right] \exp\left[\frac{2}{N_0} \int_0^T x(t)\, s(t)\, dt\right] dP_S(s) . \tag{3.7b}$$

If the signal is known exactly or completely specified, the probability for that signal, or point s, is unity, and the probability for any set of points not containing s is zero. Then the likelihood ratio becomes

$$\ell_s(x) = \exp\left[-\frac{E(s)}{N_0}\right] \exp\left[\frac{1}{N} \sum_{i=1}^{n} x_i s_i\right], \quad \text{or} \tag{3.8a}$$

$$= \exp\left[-\frac{E(s)}{N_0}\right] \exp\left[\frac{2}{N_0} \int_0^T x(t)\, s(t)\, dt\right] . \tag{3.8b}$$

Thus the general formulas (3.7a) and (3.7b) for likelihood ratio state that $\ell(x)$ is the weighted average of $\ell_s(x)$ over the set of all signals, i.e.,

$$\ell(x) = \int_R \ell_s(x)\, dP_S(s) . \tag{3.9}$$

If the distribution function $P_S(s)$ depends on various parameters such as carrier phase, signal energy, or carrier frequency, and if the distributions

in these parameters are independent,[1] the expression for likelihood ratio can be simplified somewhat. If these parameters are indicated by $r_1$, $r_2$, ..., $r_n$, and the associated probability density functions are denoted by $f_1(r_1)$, $f_2(r_2)$, ..., $f_n(r_n)$, then

$$d\, P_S(s) = f_1(r_1) \cdots f_n(r_n)\, dr_1 \cdots dr_n$$

The likelihood ratio becomes

$$\mathcal{L}(x) = \int \cdots \int \mathcal{L}_s(x)\, f_1(r_1) \cdots f_n(r_n)\, dr_1 \cdots dr_n$$

$$= \int \left[ f_n(r_n) \cdots \left[ \int f_1(r_1)\, \mathcal{L}_s(x)\, dr_1 \right] \cdots \right] dr_n \quad . \tag{3.10}$$

Thus the likelihood ratio can be found by averaging $\mathcal{L}_s(x)$ with respect to the parameters.

---

[1]Cramér, Ref. 14, p. 159.

## 4. LIKELIHOOD RATIO AND ITS DISTRIBUTION FOR SPECIAL CASES

### 4.1 Introduction

The purpose of this section is to derive expressions or approximate expressions for likelihood ratio and its distribution functions for a number of special signals in the presence of Gaussian noise. The results obtained in this section are summarized and discussed in Section 5.

### 4.2 The Case of a Signal Known Exactly

The likelihood ratio for the case when the signal is known exactly has already been presented in Section 3.3, Eq (3.8).

$$\ell(x) = \exp\left[-\frac{E}{N_0}\right] \exp\left[\frac{1}{N} \sum_{i=1}^{n} x_i s_i\right] \quad , \tag{4.1a}$$

$$\ell(x) = \exp\left[-\frac{E}{N_0}\right] \exp\left[\frac{2}{N_0} \int_0^T x(t)\, s(t)\, dt\right] \tag{4.1b}$$

As the first step in finding the distribution functions for $\ell(x)$, it is convenient to find the distribution for $\frac{1}{N} \sum x_i s_i$ when there is noise alone. Then the input $x = (x_1, x_2, \ldots, x_i)$ is due to white Gaussian noise. It can be seen from Eq (3.1) that each $x_i$ has a normal distribution with zero mean and variance $N = W N_0$ and that the $x_i$ are independent. Because the $s_i$ are constants depending on the signal to be detected, $s = (s_1, s_2, \ldots, s_n)$, each summand $\frac{2}{N}(x_i s_i)$ has a normal distribution with mean $\frac{s_i}{N}$ times the mean of $x_i$, and variance $\frac{s_i^2}{N^2}$ times the variance of $x_i$ — zero and $\frac{s_i^2}{N^2} N = \frac{s_i^2}{N}$ respectively.

Because the $x_i$ are independent, the summands $\frac{1}{N} s_i x_i$ are independent, each with normal distributions, and therefore their sum has a normal distribution with

mean the sum of the means — i.e., zero — and variance the sum of the variances,[1]

$$\sum \frac{s_i^2}{N} = \frac{2WE(s)}{N} = \frac{2E}{N_0} = 2 \times \frac{\text{Signal Energy}}{\text{Noise Power Per Unit Bandwidth}} \quad \cdot \quad (4.2)$$

The distribution for $\frac{1}{N} \sum x_i s_i$ with noise alone is thus normal with zero mean and variance $\frac{2E}{N_0}$. Recalling (4.1a)

$$\ell(x) = \exp \left[ -\frac{E}{N_0} + \frac{1}{N} \sum x_i s_i \right] \qquad (4.1a)$$

it is seen that the distribution for $\frac{1}{N} \sum x_i s_i$ can be used directly by introducing $\alpha$ defined by

$$\beta = \exp \left[ -\frac{E}{N_0} + \alpha \right] \quad , \quad \text{or } \alpha = \frac{E}{N_0} + \ell n \, \beta \qquad (4.3)$$

The inequality $\ell(x) \geq \beta$ is equivalent to $\frac{1}{N} \sum x_i s_i \geq \alpha$, and therefore

$$F_N(\beta) = \sqrt{\frac{N_0}{4\pi E}} \int_\alpha^\infty \exp \left[ -\frac{1}{2} \frac{N_0}{2E} y^2 \right] \, dy \qquad (4.4)$$

The distribution for the case of signal plus noise can be found by using Theorem 8, which states that[2]

$$dF_{SN}(\beta) = \beta \, dF_N(\beta) \quad . \qquad (4.5)$$

---

[1] Cramér, Ref. 14, p. 212.

[2] See Part I, pp. 24 and 27.

Differentiating Eq (4.4),

$$dF_N(\beta) = -\sqrt{\frac{N_o}{4\pi E}} \; \exp \left( -\frac{N_o \alpha^2}{4E} \right) d\alpha \qquad , \qquad ^1 \qquad (4.6)$$

and combining (4.3), (4.5), and (4.6),

$$dF_{SN}(\beta) = -\sqrt{\frac{N_o}{4\pi E}} \; \exp \left[ -\frac{E}{N_o} + \alpha - \frac{N_o \alpha^2}{4E} \right] d\alpha \qquad (4.7)$$

Thus

$$F_{SN}(\beta) = \sqrt{\frac{N_o}{4\pi E}} \int_\alpha^\infty \exp \left[ -\frac{N_o}{4E} \left( y - \frac{2E}{N_o} \right)^2 \right] dy \qquad (4.8)$$

In summary, $\alpha$, and therefore $\ln \beta$, has a normal distribution with signal plus noise as well as with noise alone; the variance of both distributions is $\frac{2E}{N_o}$, and the difference of the means is $\frac{2E}{N_o}$.

The receiver operating characteristic curves in Fig. 4.1 are plotted for any case in which $\ln \ell$ has a normal distribution with the same variance both with noise alone and with signal plus noise. The parameter d in this figure is equal to the square of the difference of the means, divided by the variance. These receiver operating characteristic curves apply to the case of the signal known exactly, with $d = \frac{2E}{N_o}$.

Eq (4.1b) describes what the ideal receiver should do for this case.

The essential operation in the receiver is obtaining the correlation, $\int_0^T s(t)x(t)dt$.

---

[1] The change in sign appears because the distribution functions $F_{SN}(\beta)$ and $F_N(\beta)$ are probabilities that $\ell(x)$ will lie between $\beta$ and $\infty$, not $-\infty$ and $\beta$ as is usually the case. If the density function for $F_{SN}(\beta)$ is called $g(\beta)$, then

$$\frac{dF_{SN}(\beta)}{d\beta} = -g(\beta), \text{ and } F_{SN}(\beta) = \int_\beta^\infty g(\beta) \; d\beta.$$

The other operations, multiplying by a constant, adding a constant, and taking the exponential function, can be taken care of simply in the calibration of the receiver output. Electronic means of obtaining cross correlation have been developed recently.[1]

If the form of the signal is simple, there is a simple way to obtain this cross correlation.[2] Suppose $h(t)$ is the impulse response of a filter. The response $e_o(t)$ of the filter to a voltage $x(t)$ is[3]

$$e_o(t) = \int_{-\infty}^{t} x(\tau) \, h(t-\tau) \, d\tau \qquad (4.9)$$

If a filter can be synthesized so that

$$h(t) = s(T-t) \qquad 0 \le t \le T$$
$$h(t) = 0 \qquad \text{otherwise,} \qquad (4.10)$$

then

$$e_o(T) = \int_{0}^{T} x(\tau) \, s(\tau) \, d\tau \quad , \qquad (4.11)$$

so that the response of this filter at time $T$ is the cross correlation required. Thus, the ideal receiver consists simply of a filter and amplifiers.

It should be noted that this filter is the same, except for a constant factor, as that specified when one asks for the filter which maximizes peak signal to average noise power ratio.[4]

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[1] Harrington and Rogers, Ref. 16; Harting and Meade, Ref. 17; Lee, Cheatham, and Wiesner, Ref. 18; Levin and Reintzes, Ref. 19.

[2] This appears to be due to Woodward. See Woodward, Ref. 5, and Woodward and Davies, Ref. 3.

[3] S. Goldman, Transformation Calculus and Electrical Transients, Prentice Hall, New York, 1949, p. 112.

[4] Lawson and Uhlenbeck, Ref. 1, p. 206; North, Ref. 11.

FIG. 4.1

RECEIVER OPERATING CHARACTERISTIC.

$\ln l$ IS A NORMAL DEVIATE WITH $\sigma_N^2 = \sigma_{SN}^2$, $(M_{SN} - M_N)^2 = d \, \sigma_N^2$.

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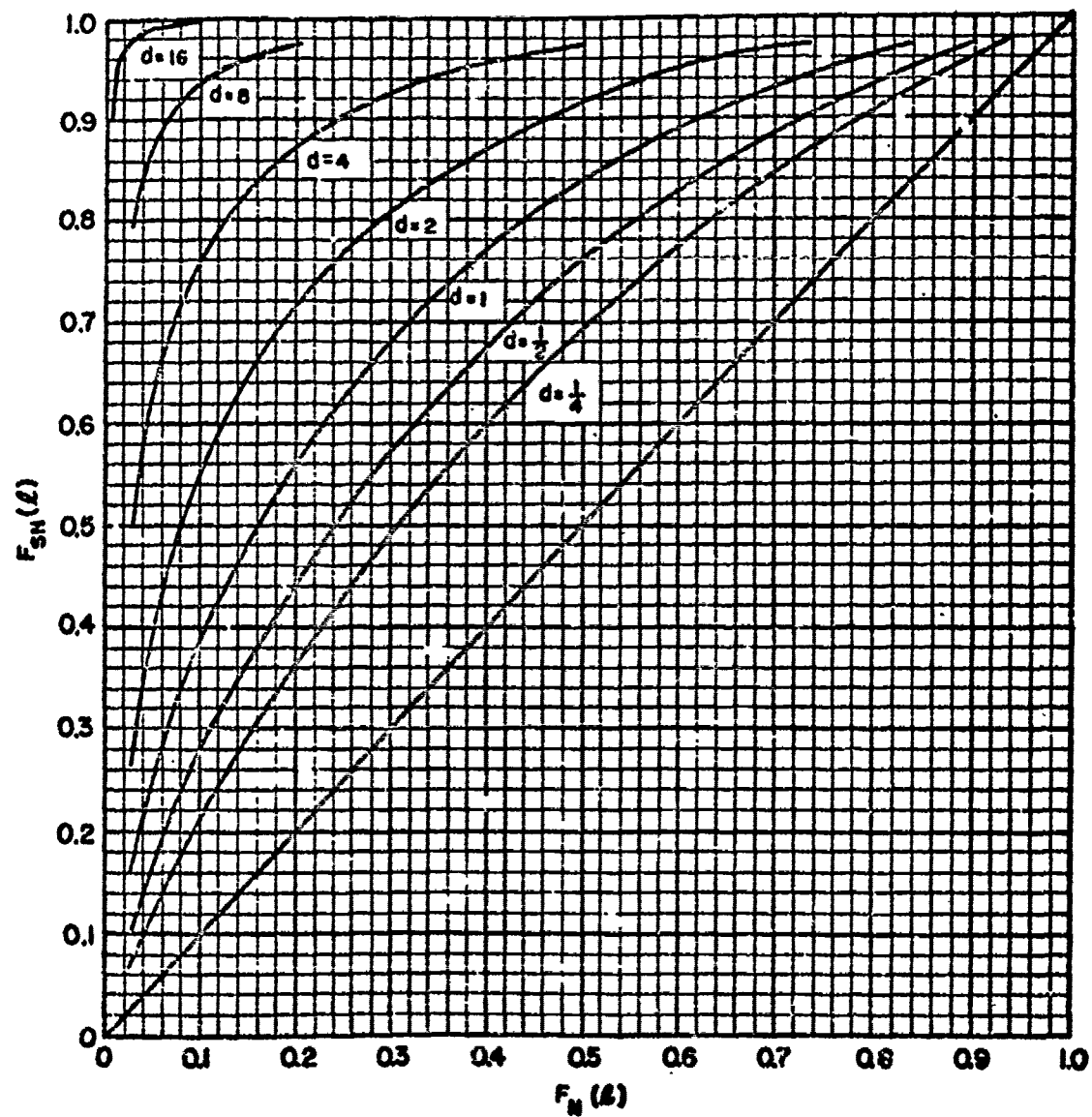
FIG. 4.2
RECEIVER OPERATING LEVEL.
$\ln \ell$ IS A NORMAL DEVIATE WITH
$\sigma_N^2 = \sigma_{SN}^2$, $(M_{SN} - M_N)^2 = d\ \sigma_N^2$.
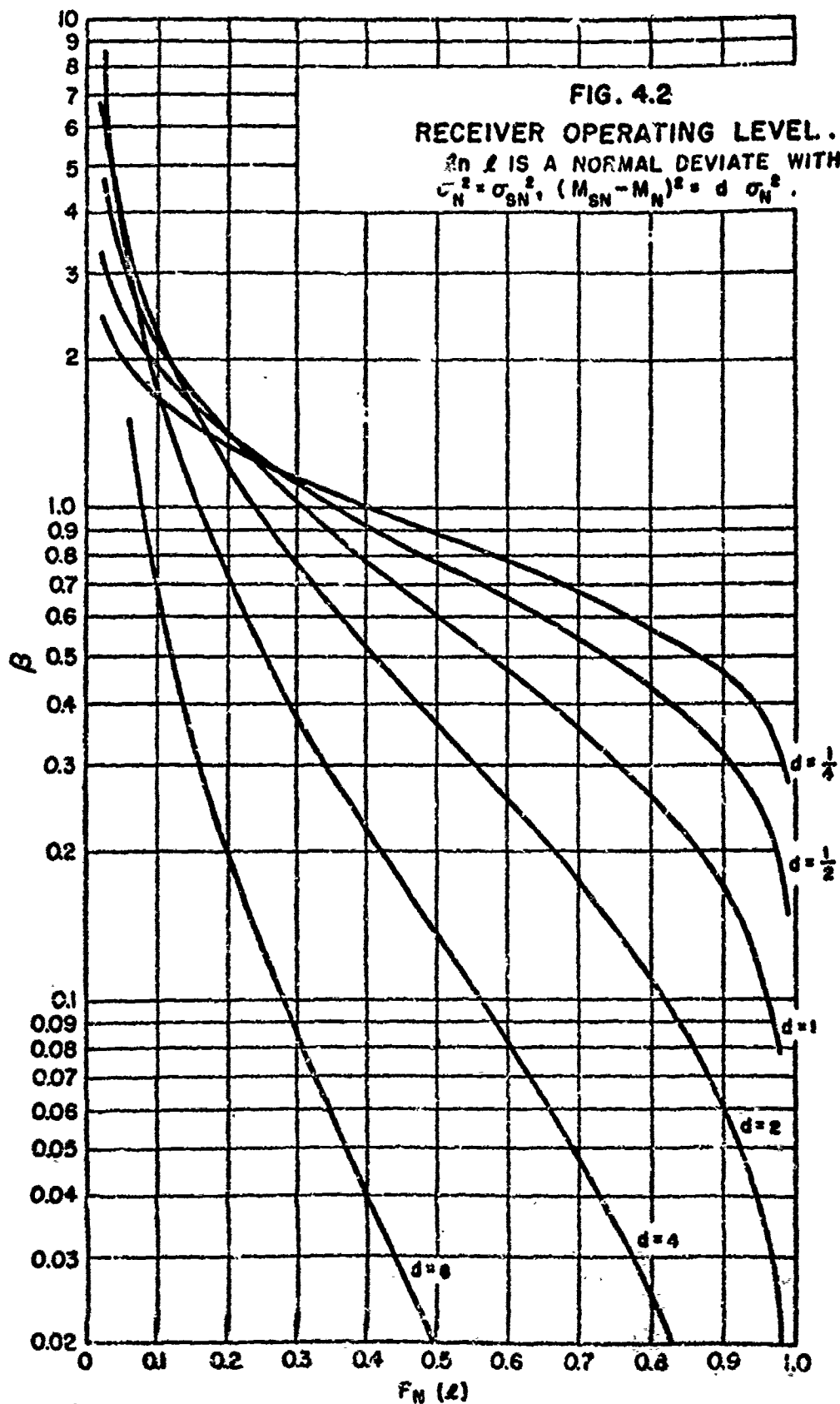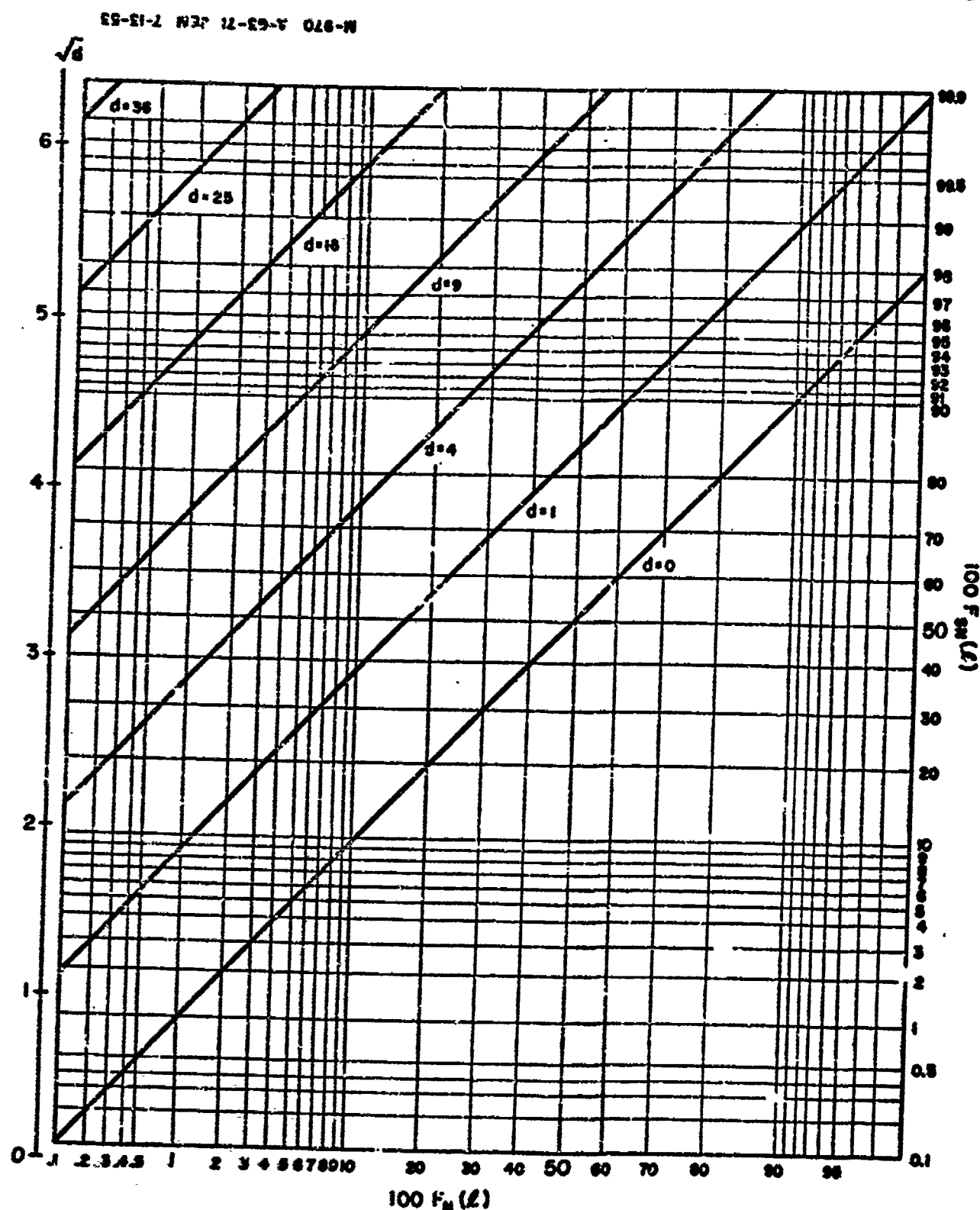
FIG. 4.3

RECEIVER OPERATING CHARACTERISTIC.

$\ell$, $\ell$ IS A NORMAL DEVIATE, $\sigma_{SN}^2 = \sigma_N^2$, $(M_{SN} - M_N)^2 = d\,\sigma_N^2$
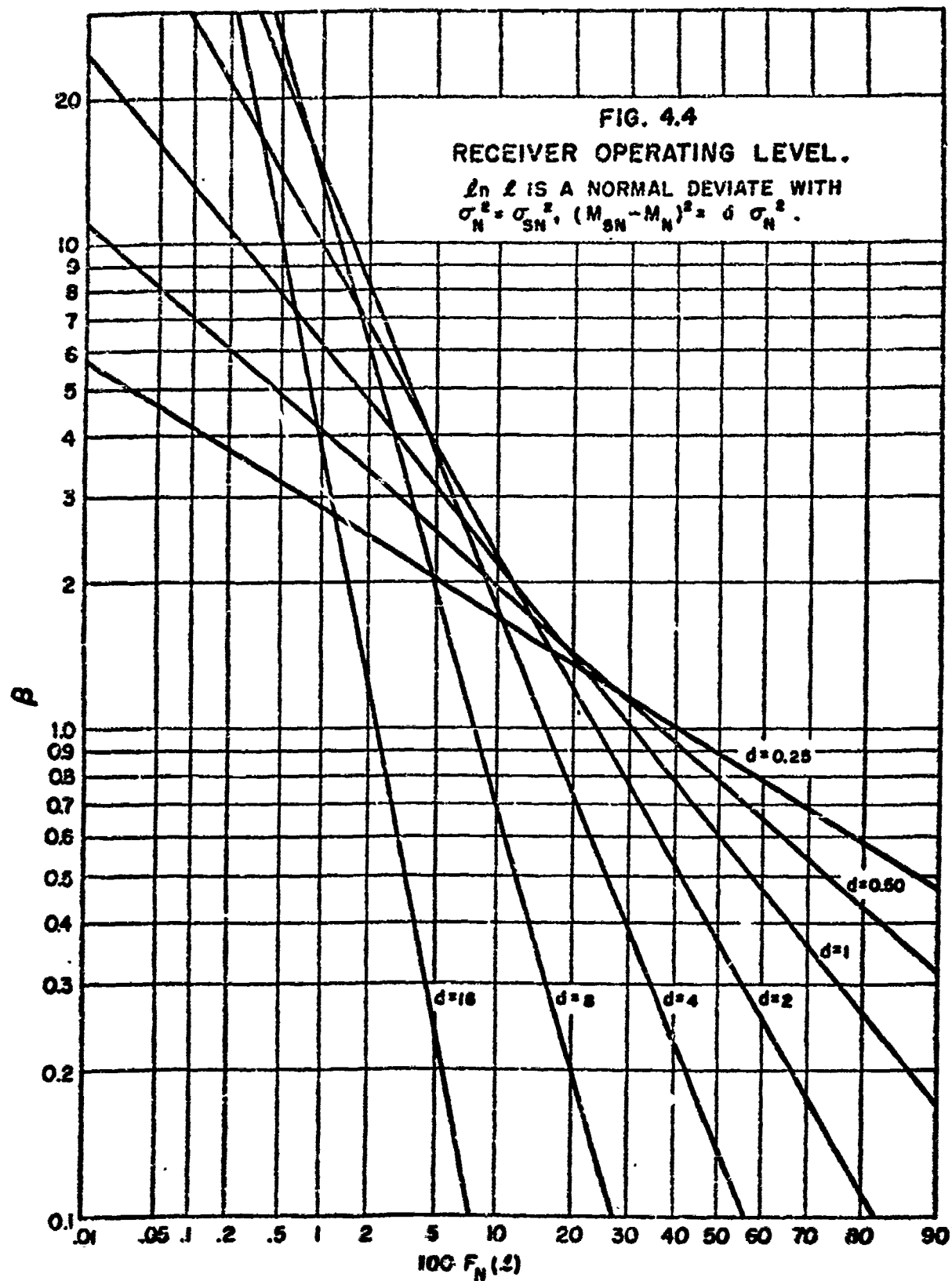
15

FIG. 4.4
RECEIVER OPERATING LEVEL.
$\ell n \, \ell$ IS A NORMAL DEVIATE WITH
$\sigma_N^2 = \sigma_{SN}^2$, $(M_{SN} - M_N)^2 = d \, \sigma_N^2$.

### 4.3 Signal Known Except for Carrier Phase

The signal ensemble considered in this section consists of all signals which differ from a given amplitude and frequency modulated signal only in their carrier phase, and all carrier phases are assumed equally likely.

$$s(t) = f(t) \cos\left(\omega t + \phi(t) - \theta\right) . \tag{4.12}$$

Since the unknown phase angle $\theta$ has a uniform distribution,

$$dP_S(\theta) = \frac{1}{2\pi} d\theta. \tag{4.13}$$

The likelihood ratio can be found by applying Eq (3.7), and since the signal energy $E(s)$ is the same for all values of carrier phase $\theta$,[1]

$$\ell(x) = \exp\left[-\frac{E}{N_o}\right] \int_R \exp\left[\frac{1}{N}\sum x_i s_i\right] dP_S(s) \tag{4.14}$$

Expanding $s$ into the coefficients of $\cos\theta$ and $\sin\theta$ will be helpful:[2]

$$s(t) = f(t) \cos\left(\omega t + \phi(t)\right)\cos\theta + f(t) \sin\left(\omega t + \phi(t)\right)\sin\theta , \tag{4.15}$$

and

$$\frac{1}{N}\sum x_i s_i = \cos\theta \frac{1}{N}\sum x_i f(t_i) \cos\left(\omega t_i + \phi(t_i)\right)$$

$$+ \sin\theta \frac{1}{N}\sum x_i f(t_i) \sin\left(\omega t_i + \phi(t_i)\right) \tag{4.16}$$

Because we wish to integrate with respect to $\theta$ to find the likelihood ratio, it is easiest to introduce parameters similar to polar coordinates $(r, \theta_o)$ such that

---

[1] For this to be rigorously true, it is sufficient that the signal be time limited and have its line spectrum zero at zero frequency and at all frequencies equal to or greater than $\frac{2\omega}{2\pi}$ .

[2] $t_i$ denotes the $i^{th}$ sample point, i.e., $t_i = \frac{i}{2W}$ .

$$\frac{1}{N} r \cos \theta_0 = \frac{1}{N} \sum x_i f(t_i) \cos \left( \omega t_i + \phi(t_i) \right)$$

$$\frac{1}{N} r \sin \theta_0 = \frac{1}{N} \sum_0 x_i f(t_i) \sin \left( \omega t_i + \phi(t_i) \right) \tag{4.17}$$

and therefore

$$\frac{1}{N} \sum x_i s_i = \frac{r}{N} \cos (\theta - \theta_0) \qquad . \tag{4.18}$$

Using this form the likelihood ratio becomes

$$\ell(x) = \exp \left[ -\frac{E}{N_0} \right] \int_0^{2\pi} \exp \left[ \frac{r}{N} \cos (\theta - \theta_0) \right] \frac{d\theta}{2\pi}$$

$$= \exp \left[ -\frac{E}{N_0} \right] I_0 \left( \frac{r}{N} \right) \tag{4.19}$$

where $I_0$ is the Bessel function of zero order and pure imaginary argument.

$I_0$ is a strictly monotone increasing function, and therefore the likelihood ratio will be greater than a value $\beta$ if and only if $\frac{r}{N}$ is greater than some value corresponding to $\beta$. The quantity $r$ is defined by the Eq (4.17); $\frac{r}{N}$ is the square root of the sums of the squares of the right-hand sides. The probability that $\frac{r}{N}$ will exceed any certain value can be computed by observing that each of the right-hand sides is $\frac{N_0}{2}$ times the cross correlation of $x(t)$ with a fixed signal, either $f(t) \cos \left[ \omega t + \phi(t) \right]$ or $f(t) \sin \left[ \omega t + \phi(t) \right]$. Therefore, the distribution of each can be found in the same manner as the distribution of $\frac{1}{N} \sum x_i s_i$ was found for the case of the signal known exactly,[1] and both $\frac{r}{N} \cos \theta_0$ and $\frac{r}{N} \sin \theta_0$ have normal distributions with zero mean and variance $\frac{2E}{N_0}$. Furthermore, $f(t) \cos \left( \omega t + \phi(t) \right)$ and $f(t) \sin \left( \omega t + \phi(t) \right)$ are out of phase, or orthogonal, and therefore $r \cos \theta_0$ and $r \sin \theta_0$ have independent distributions.[2]

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[1] See page 9.   [2] See footnote 1, p. 17.

Because $\frac{r}{N} = \sqrt{\left(\frac{r}{N}\cos\theta_o\right)^2 + \left(\frac{r}{N}\sin\theta_o\right)^2}$ , the probability that $\frac{r}{N}$ will exceed any fixed value is given by the well-known chi-square distribution for two degrees of freedom, $K_2(\alpha^2)$.[1] The proper normalization yielding zero mean and unit variance requires that the variable be $\frac{r}{N}\sqrt{\frac{N_o}{2E(s)}}$, that is

$$P_N\left(\frac{r}{N}\sqrt{\frac{N_o}{2E}} \geq \alpha\right) = K_2(\alpha^2) = \exp\left[-\frac{\alpha^2}{2}\right]. \qquad (4.20)$$

If $\alpha$ is defined by the equation

$$\beta = \exp\left[-\frac{E}{N_o}\right] I_o\left(\sqrt{\frac{2E}{N_o}}\,\alpha\right), \qquad (4.21)$$

the distribution for $\ell(x)$ in the presence of noise alone is in the simple form

$$F_N(\beta) = \exp\left[-\frac{\alpha^2}{2}\right]. \qquad (4.22)$$

Using Theorem 8 of Section 2, namely

$$\beta\, dF_N(\beta) = dF_{SN}(\beta) \qquad (4.23)$$

but making use of the parameter $\alpha$, we form first

$$dF_N(\beta) = -\alpha \exp\left[-\frac{\alpha^2}{2}\right] d\alpha , \qquad (4.24)$$

and hence

$$dF_{SN}(\beta) = -\exp\left[-\frac{E}{N_o}\right]\alpha \exp\left[-\frac{\alpha^2}{2}\right] I_o\left(\sqrt{\frac{2E}{N_o}}\,\alpha\right) d\alpha \qquad (4.25)$$

Integrate from $\alpha$ to infinity.

$$F_{SN}(\beta) = \exp\left[-\frac{E}{N_o}\right]\int_\alpha^\infty \alpha \exp\left[-\frac{\alpha^2}{2}\right] I_o\left(\sqrt{\frac{2E}{N_o}}\,\alpha\right) d\alpha . \qquad (4.26)$$

---

[1]Cramér, Ref. 14, p. 233, or Hoel, P. G., **Introduction to Mathematical Statistics**, Wiley, 1947, p. 134.

Eqs (4.22) and (4.26) yield the receiver operating characteristic in parametric form, and Eq (4.21) gives the associated operating levels.[1] These are graphed in Fig. 4.5 for the same values of signal energy to noise per unit bandwidth ratio as were used when the phase angle was known exactly, Fig. 4.1, so that the effect of knowing the phase can be easily seen.

If the signal is sufficiently simple so that a filter could be synthesized to match the expected signal for a given carrier phase $\theta$ as in the case of a signal known exactly, then there is a simple way to design a receiver to obtain likelihood ratio. For simplicity let us consider only amplitude modulated signals $\left(\phi(t) \doteq 0\right)$ in Eq. (4.12)). Let us also choose $\theta = 0$. (Any phase could have been chosen.) Then the filter has impulse response

$$h(t) = f(T-t) \cos\left[\omega(T-t)\right] \qquad 0 \le t \le T$$
$$= 0 \qquad\qquad \text{otherwise.} \qquad (4.27)$$

The output of the filter in response to $x(t)$ is then

$$e_o(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{t-T}^{t} x(\tau)\, f(\tau+T-t) \cos\omega(\tau+T-t)\, d\tau$$

$$= \cos\omega(T-t) \int_{t-T}^{t} x(\tau)\, f(\tau+T-t) \cos\omega\tau\, d\tau$$

$$- \sin\omega(T-t) \int_{t-T}^{t} x(\tau)\, f(\tau+T-t) \sin\omega\tau\, d\tau \qquad . \qquad (4.28)$$

---

[1] Graphs of values of the integral (4.26) along with approximate expressions for small and for large values of $\alpha$ appear in Rice, Ref. 20. Tables of this function have been compiled by J. I. Marcum in an unpublished report of the Rand Corporation, "Table of Q-Functions," Project Rand Report RM-399.
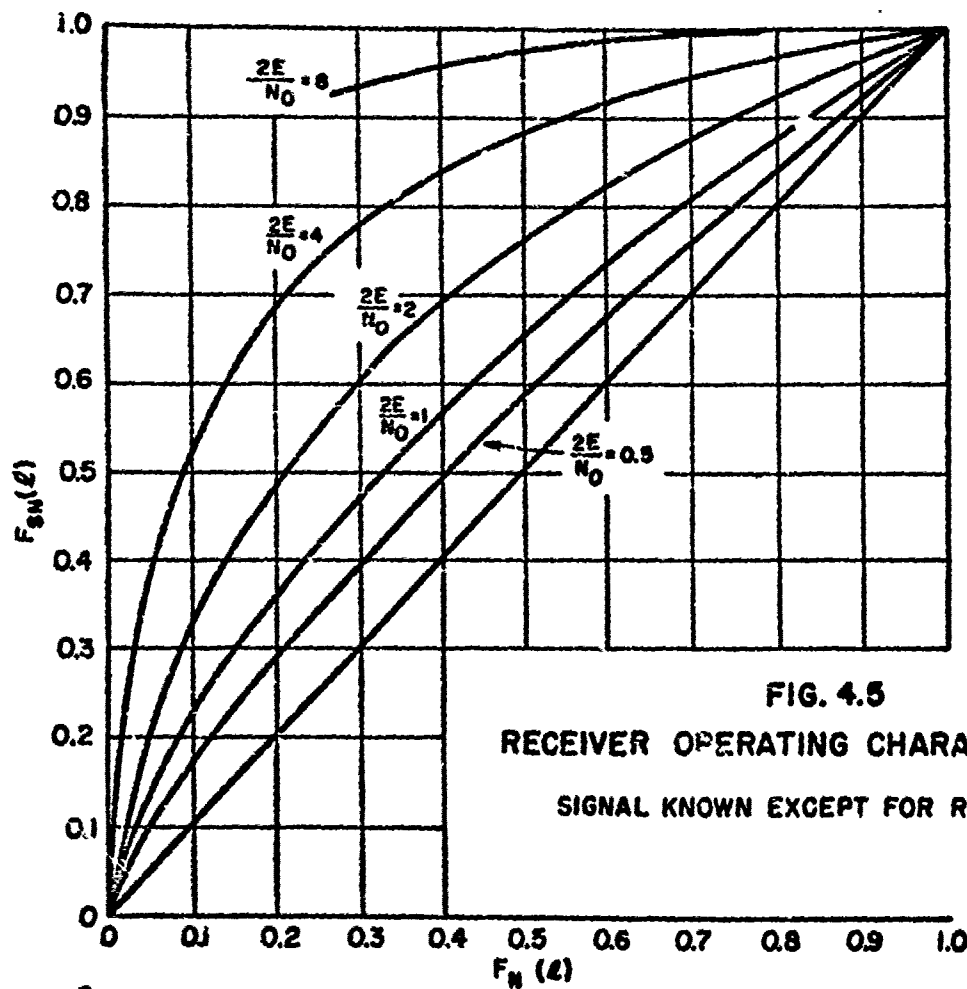
FIG. 4.5

RECEIVER OPERATING CHARACTERISTIC.

SIGNAL KNOWN EXCEPT FOR R.F. PHASE.

21
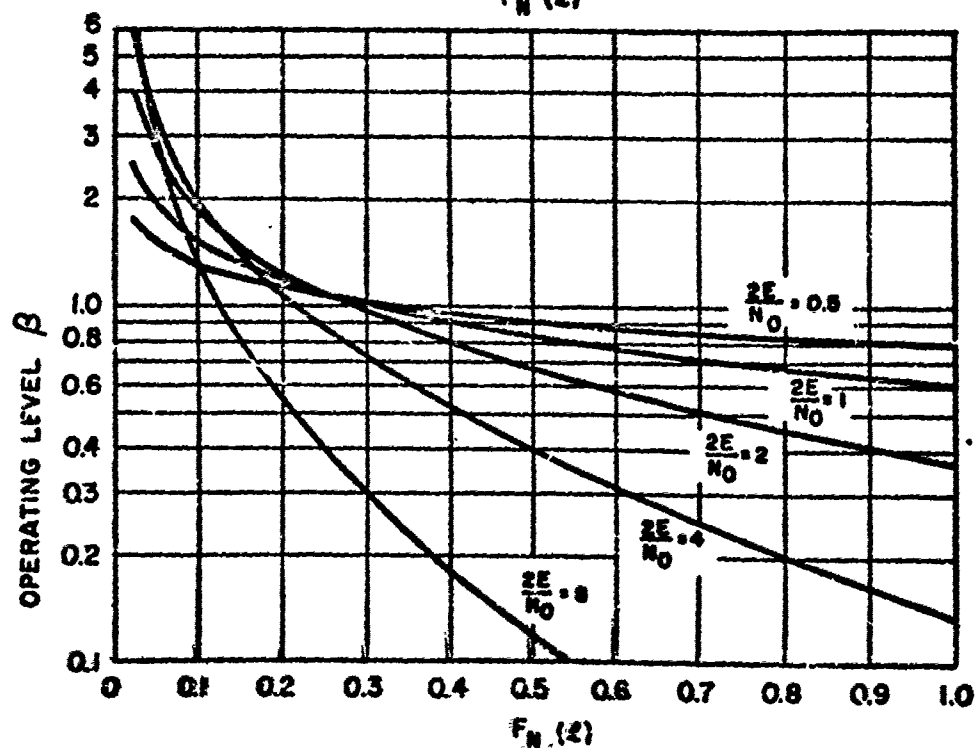
The envelope of the filter output will be the square root of the sum of the squares of the integrals,[1] and the envelope at time T will be proportional to $\frac{r}{N}$, since

$$\left(\frac{r}{2N}\right)^2 = \left[\int_0^T x(\tau)\, f(\tau)\, \cos \omega \tau\, d\tau\right]^2 + \left[\int_0^T x(\tau)\, f(\tau)\, \sin \omega \tau\, d\tau\right]^2. \quad (4.29)$$

= Square of the envelope, at time T, of $e_0(t)$.

If the input $x(t)$ passes through the filter with an impulse response given by Eq (4.27), then through a linear detector, the output will be $\frac{N_0}{2} \frac{r}{N}$ at time T. Because the likelihood ratio, Eq (4.19), is a known monotone function of $\frac{r}{N}$, the output can be calibrated to read the likelihood ratio of the input.

## 4.4  Signal Consisting of a Sample of White Gaussian Noise

Suppose the values of the signal voltage at the sample points are independent Gaussian random variables with zero mean and variance S, the signal power. The probability density due to signal plus noise is also Gaussian, since signal plus noise is the sum of two Gaussian random variables:[2]

$$f_{SN}(x) = \left(\frac{1}{2\pi(N+S)}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2} \frac{1}{N+S} \sum x_i^2\right]. \quad (4.30)$$

The likelihood ratio is

$$\ell(x) = \left(\frac{N}{N+S}\right)^{\frac{n}{2}} \exp\left[\frac{1}{2} \frac{1}{N} \sum x_i^2 - \frac{1}{2} \frac{1}{N+S} \sum x_i^2\right] \quad (4.31)$$

---

[1] If the line spectrum of $x(t)$ is zero at zero frequency and at all frequencies equal to or greater than $\frac{2\omega}{2\pi}$, then it can be shown that these integrals contain no frequencies as high as $\frac{\omega}{2\pi}$.

[2] Cramér, Ref. 14, p. 212.

In solving for the distribution functions for $\ell$, it is convenient to introduce the parameter $\alpha$, defined by the equation

$$\beta = \left(\frac{N}{N+S}\right)^{\frac{n}{2}} \exp\left(\frac{S}{N+S} \frac{\alpha^2}{2}\right) \quad . \tag{4.32}$$

Then the condition $\ell(x) \geq \beta$ is equivalent to the condition that $\frac{1}{N} \sum x_i \geq \alpha^2$. In the presence of noise alone the random variables $\left(\frac{x_i}{\sqrt{N}}\right)$ have zero mean and unit variance, and they are independent. Therefore, the probability that the sum of the squares of these variables will exceed $\alpha^2$ is the chi-square distribution with n degrees of freedom,[1] i.e.,

$$F_N(\beta) = K_n(\alpha^2) \quad . \tag{4.33}$$

Similarly, in the presence of signal plus noise the random variables $\left(\frac{x_i}{\sqrt{N+S}}\right)$ have zero mean and unit variance. The condition $\frac{1}{N} \sum x_i^2 \geq \alpha^2$ is the same as requiring that $\frac{1}{N+S} \sum x_i^2 \geq \frac{N}{N+S} \alpha^2$, and again making use of the chi-square distribution,

$$F_{SN}(\beta) = K_n\left(\frac{N}{N+S} \alpha^2\right) . \tag{4.34}$$

Receiver operating characteristic curves are presented in Figs. 4.6 and 4.7 for four possible choices of n ($10^2$, $10^3$, $10^4$, $10^5$), and in each case for three values of signal to noise ratio three db apart.

For large values of n, the chi-square distribution is approximately normal over the center portion; more precisely,[2] for $\alpha^2 >> 0$

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[1] Cramér, Ref. 14, p. 233. Tables of $K_n(\alpha^2)$ can be found in most books on statistics. Extensive tables are listed in the bibliography of Ref. 14, p. 570.

[2] P. G. Hoel, Introduction to Mathematical Statistics, New York: Wiley, 1947, p. 246.
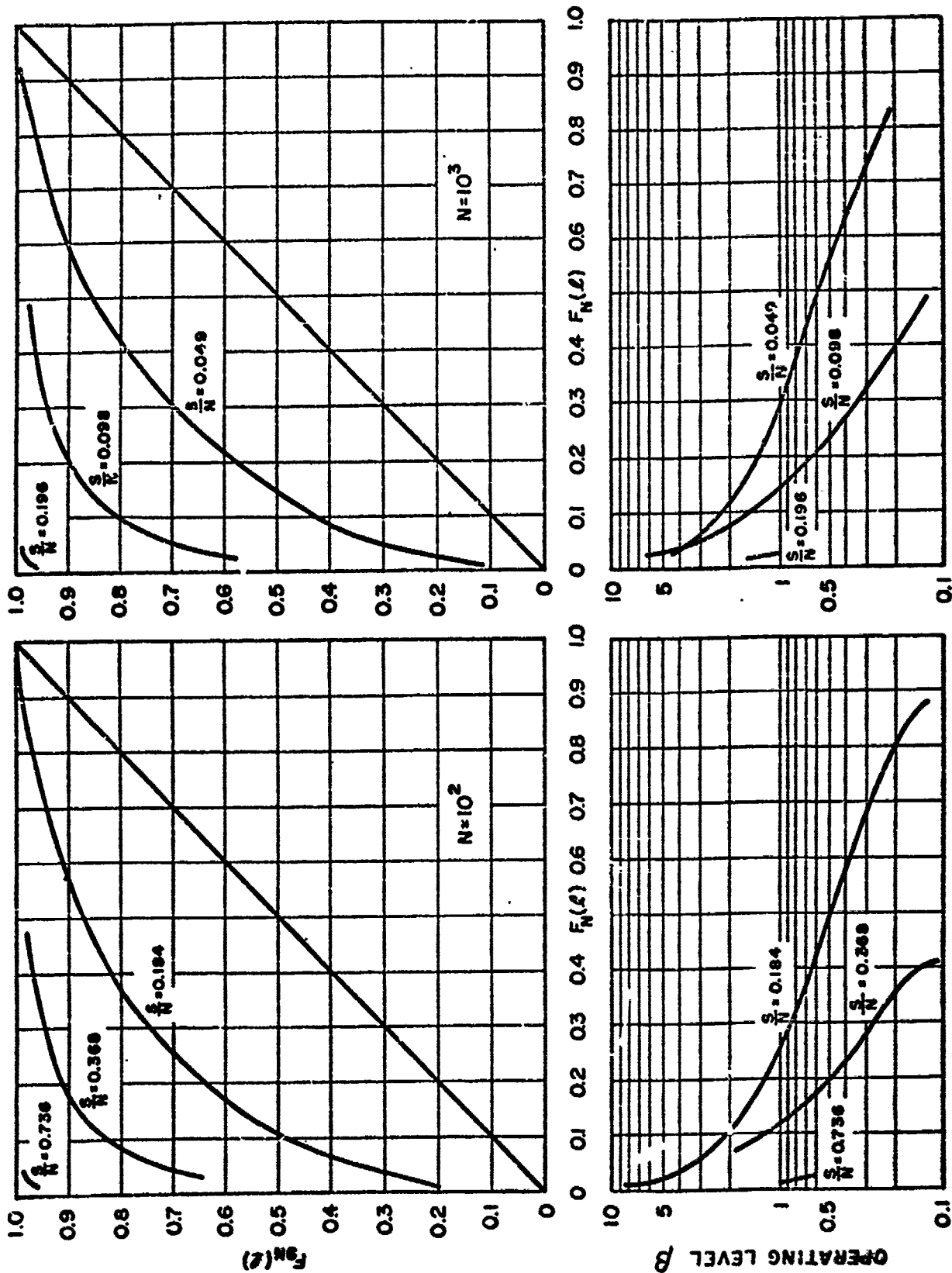
FIG. 4.6. RECEIVER OPERATING CHARACTERISTIC.
SIGNAL A SAMPLE OF WHITE GAUSSIAN NOISE.
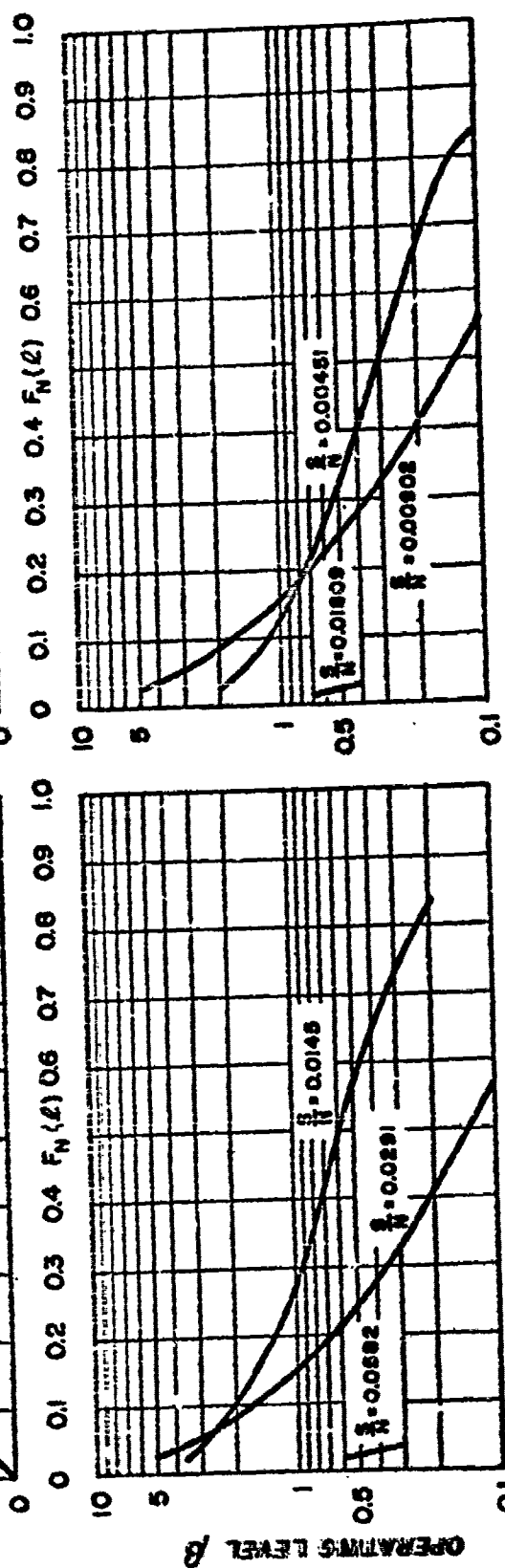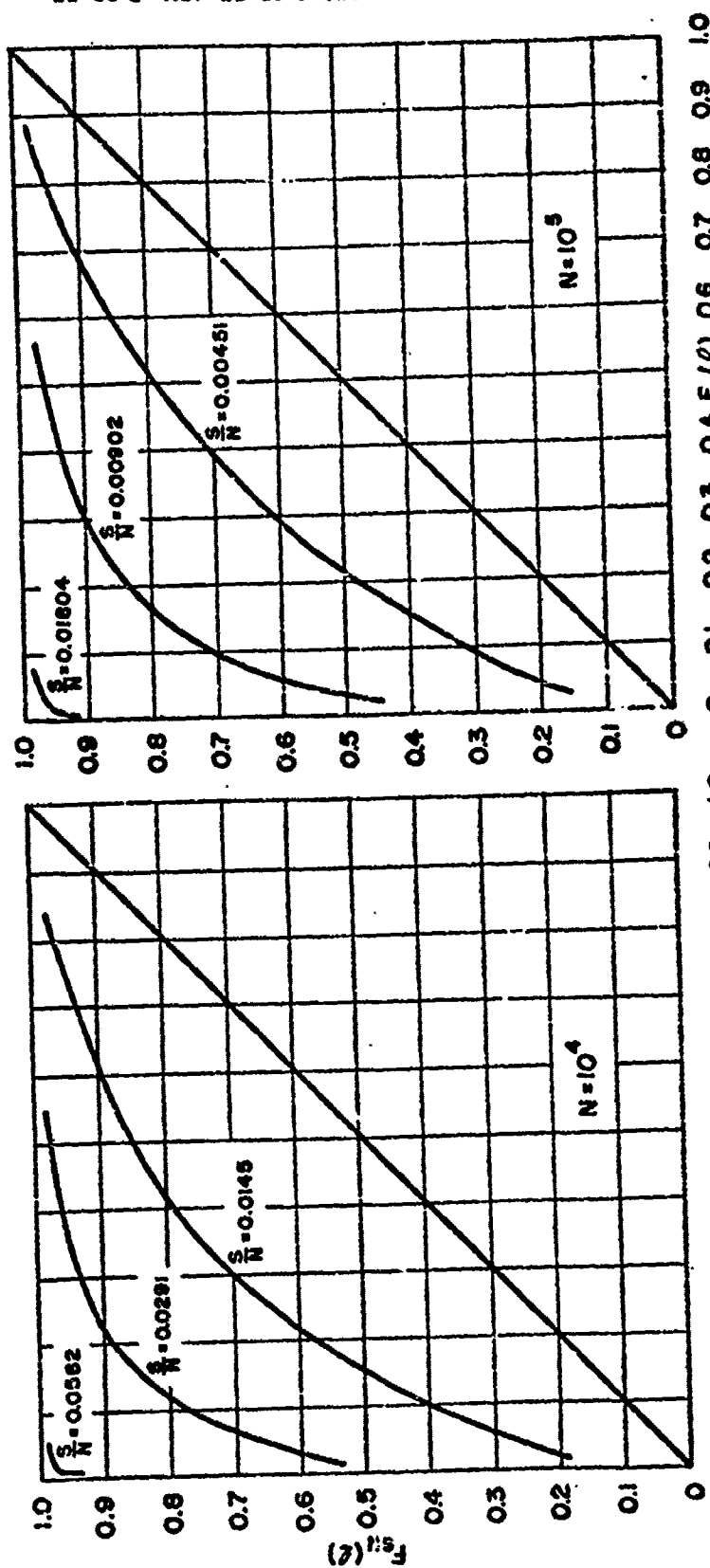
24

FIG. 4.7. RECEIVER OPERATING CHARACTERISTIC.
SIGNAL A SAMPLE OF WHITE GAUSSIAN NOISE.

25

$$K_n(\alpha^2) \approx \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2}\,\alpha^2 - \sqrt{2n-1}}^{\infty} \exp\left[-\frac{1}{2}y^2\right] dy \qquad (4.35)$$

and

$$K_n\left(\frac{N}{N+S}\,\alpha^2\right) \approx \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2N\alpha^2}{N+S}} - \sqrt{2n-1}}^{\infty} \exp\left[-\frac{1}{2}y^2\right] dy$$

If the signal energy is small compared to that of the noise, $\sqrt{\frac{N}{N+S}}$ is nearly unity and both distributions have nearly the same variance. Then Fig. 4.1 applies to this case too, with the value of d given by

$$d = (2n-1)\left(1 - \sqrt{\frac{N}{N+S}}\right)^2 \quad . \qquad (4.37)$$

For these small signal to noise ratios and large samples, there is simple relation between signal to noise ratio, the number of samples, and the detection index d.

$$1 - \sqrt{\frac{N}{N+S}} \approx \frac{1}{2}\frac{S}{N} \qquad \text{for } \frac{S}{N} \ll 1 \quad ,$$

$$d \approx \frac{nS^2}{2N^2} \qquad (4.38)$$

Two signal to noise ratios, $\left(\frac{S}{N}\right)_1$ and $\left(\frac{S}{N}\right)_2$, will have approximately the same operating characteristic if the corresponding numbers of sample points, $n_1$ and $n_2$, satisfy

$$\frac{n_1}{n_2} = \frac{\left(\frac{S}{N}\right)_1^2}{\left(\frac{S}{N}\right)_2^2}$$

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This can be verified for the three curves of Fig. 4.7 for $n = 10^5$, compared with Fig. 4.1 for $d = 1, 4, 16$.

The receiver specified is any device that produces the likelihood ratio of its input,

$$\ell(x) = \left(\frac{N}{N+S}\right)^{\frac{n}{2}} \exp\left[\frac{S}{N+S} \frac{1}{N} \sum x_i^2\right] . \tag{4.31}$$

An energy detector has as its output

$$e_0(t) = \int_0^T \left[x(t)\right]^2 dt = \frac{1}{2W} \sum x_i^2 \tag{4.40}$$

and this receiver can be calibrated so that its output at the end of the observation time, $e_0(t)$, will be read as

$$\ell(x) = \left(\frac{N}{N+S}\right)^{\frac{n}{2}} \exp\left[\frac{S}{N+S} \frac{e_0(T)}{N_0}\right] \tag{4.41}$$

## 4.5   Video Design of a Broad Band Receiver

The problem considered in this section is represented schematically in Fig. 4.8. The signals and noise are assumed to have passed through a band
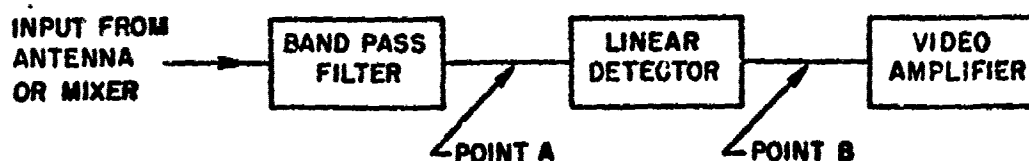


FIG. 4.8

BLOCK DIAGRAM OF A BROAD BAND RECEIVER.

pass filter, and at the output of the filter, point A on the diagram, they are assumed to be limited in spectrum to a band of width $W$ and center frequency

$\frac{\omega}{2\pi} > \frac{W}{2}$ . The noise is assumed to be Gaussian noise with a uniform spectrum over the band. The signals and noise then pass through a linear detector. The output of the detector is the envelope of the signals and noise as they appeared at point A; all knowledge of the phase of the receiver input is lost at point B. The signals and noise as they appear at point B are considered receiver inputs, and the theory of signal detectability is applied to these video inputs to ascertain the best video design and the performance of such a system. The mathematical description of the signals and noise will be given for the signals and noise as they appear at point A. The envelope functions, which appear at point B, will be derived, and the likelihood ratio and its distribution will be found for these envelope functions.

The only case which will be considered here is the case in which the amplitude of the signal as it would appear at point A is a known function of time.

Any function at point A will be band limited to a band of width W and center frequency $\frac{\omega}{2\pi} > \frac{W}{2}$ . Then the alternate form of the sampling theorem can be used.[1] Any such function $f(t)$ can be expanded as follows:

$$f(t) = x(t) \cos \omega t + y(t) \sin \omega t \tag{4.42}$$

where $x(t)$ and $y(t)$ are band limited to frequencies no higher than $\frac{W}{2}$ , and hence can themselves be expanded by the sampling theorem:

$$f(t) = \sum_i \left[ x\left(\frac{i}{W}\right) \psi_i(t) \cos \omega t + y\left(\frac{i}{W}\right) \psi_i(t) \sin \omega t \right]. \tag{4.43}$$

The function can be thought of as a point in a space of $n = 2WT$ dimensions with coordinates $x\left(\frac{i}{W}\right) = x_i$ and $y\left(\frac{i}{W}\right) = y_i$ . This is a rectangular coordinate

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[1] See Appendix D.

system, since the family of functions $\psi_i(t) \cos \omega t$ and $\psi_i(t) \sin \omega t$ form an orthogonal system.

The amplitude of the function $f(t)$ is

$$r(t) = \sqrt{\left[x(t)\right]^2 + \left[y(t)\right]^2} \qquad (4.44)$$

and thus the amplitude at the $i^{th}$ sampling point is

$$r\left(\frac{i}{W}\right) = r_i = \sqrt{x_i^2 + y_i^2} \qquad (4.45)$$

The angle

$$\theta_i = \arctan \frac{y_i}{x_i} = \arccos \frac{x_i}{r_i} \qquad (4.46)$$

might be considered the phase of $f(t)$ at the $i^{th}$ sampling point. The function $f(t)$ then might be described by giving the $r_i$ and $\theta_i$ rather than the $x_i$ and $y_i$. The $r_i$ and $\theta_i$ are sample values of amplitude and phase, and form a sort of polar coordinate system in the space associated with the set of functions.

Let us denote by $x_i$, $y_i$, or $r_i$, $\theta_i$, the coordinates or sample values for a receiver input after the filter (i.e., at point A in Fig. 4.8). Let $a_i$, $b_i$, or $f_i$, $\phi_i$ denote the coordinates for the signal as it would appear at point A if there were no noise. The envelope of the signal, hence the coordinates $f_i$, are assumed known. Let us denote by $F_S(\phi_1, \phi_2, \ldots, \phi_{\frac{n}{2}})$ the distribution function of the phase coordinates $\phi_i$. The probability density function for the coordinates $x_i$, $y_i$ when there is white Gaussian noise and no signal is

$$f_N(x, y) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2N}\left(\sum_{i=1}^{n/2} x_i^2 + \sum_{i=1}^{n/2} y_i^2\right)\right] \qquad (4.47)$$

and for signal plus noise

$$f_{SN}(x, y) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \int_R \exp\left[-\frac{1}{2N}\left(\sum_{i=1}^{n/2}(x_i-a_i)^2 + \sum_{i=1}^{n/2}(y_i-b_i)^2\right)\right] dP_S(a_i b_i) \quad .$$

(4.48)

Changing to the polar coordinates,

$$f_N(r, \theta) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \prod_{i=1}^{n/2} r_i \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2} r_i^2\right] \quad ,$$

(4.49)

and

$$f_{SN}(r,\theta) = \left(\frac{1}{2\pi N}\right)^{\frac{n}{2}} \prod_{i=1}^{n/2} r_i \int_R \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2}\left\{r_i^2+r_i^2-2r_ir_i \cos(\theta_i-\phi_i)\right\}\right]$$

$$dF_S\left(\phi_1, \cdots, \phi_{\frac{n}{2}}\right) \quad . \quad (4.50)$$

The factors $\prod_{i=1}^{2\pi} r_i$ are introduced because they are the Jacobian of the transformation from rectangular to polar coordinates.[1,2]

The probability density function for r alone, i.e., the density function for the output of the detector, is obtained by simply integrating the density functions for r and θ with respect to θ.[3]

$$f_N(r) = \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} f_N(r_i, \theta_i) \, d\theta_1 \, d\theta_2 \cdots d\theta_{\frac{n}{2}}$$

$$= \left(\frac{1}{N}\right)^{\frac{n}{2}} \prod_{i=1}^{n/2} r_i \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2} r_i^2\right]$$

(4.51)

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[1] Cramér, Ref. 14, page 292.

[2] For example, in two dimensions, $f_N(x, y)dx \, dy = f_N(r, \theta) \, r \, dr \, d\theta$.

[3] Cramér, Ref. 14, page 291.

and

$$f_{SN}(r) = \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f_{SN}(r_i, \theta_i) \, d\theta_1 \, d\theta_2 \cdots d\theta_{\frac{n}{2}}$$

$$= \left(\frac{1}{N}\right)^{\frac{n}{2}} \int_R \prod_{i=1}^{n/2} r_i \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2}(r_i^2 + f_i^2)\right] \prod_{i=1}^{n/2} I_0\left(\frac{r_i f_i}{N}\right) w(\phi_1, \phi_2 \cdots \phi_{\frac{n}{2}})$$

$$= \left(\frac{1}{N}\right)^{\frac{n}{2}} \prod_{i=1}^{n/2} r_i \, I_0\left(\frac{r_i f_i}{N}\right) \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2}(r_i^2 + f_i^2)\right] \qquad (4.52)$$

Notice that the probability density for r is completely independent of the distribution which the $\phi_i$ had; all information about the phase of the signals has been lost.

The likelihood ratio for a video input is

$$\ell(r) = \frac{f_{SN}(r)}{f_N(r)} = \exp\left[-\frac{1}{2N}\sum_{i=1}^{n/2} f_i^2\right] \prod_{i=1}^{n/2} I_0\left(\frac{r_i f_i}{N}\right). \qquad (4.53)$$

Again it is more convenient to work with the logarithm of the likelihood ratio.

$$\frac{1}{2N}\sum_{i=1}^{n/2} f_i^2 = \frac{W}{2N}\int [f(t)]^2 \, dt = \frac{E}{N_0}, \text{ and} \qquad (4.54)$$

$$\ln \ell(r) = -\frac{E}{N_0} + \sum_{i=1}^{n/2} \ln I_0\left(\frac{r_i f_i}{N}\right). \qquad (4.55)$$

which is approximately

$$\ln \ell(r(t)) = -\frac{E}{N_0} + W \int_0^T \ln I_0\left(\frac{r(t) \, f(t)}{N}\right) dt. \qquad (4.56)$$

The function $\ell n\, I_o(x)$ is plotted as a function of x in Fig. 4.9. This function is very nearly the parabola $\frac{x^2}{4}$ for small values of x and is approximately linear for large values of x. Thus, the expression for likelihood ratio might be approximated by

$$\ell n\, \ell\big(r(t)\big) = -\frac{E}{N_o} + \frac{W}{4N^2} \int_0^T \big[r(t)\big]^2 \big[f(t)\big]^2\, dt \qquad (4.57)$$

for small signals, and by

$$\ell n\, \ell\big(r(t)\big) = C_1 + C_2 \int_0^T r(t)\, f(t)\, dt \qquad (4.58)$$

for large signals, where $C_1$ and $C_2$ are chosen to approximate $\ell n\, I_o$ best in the desired range.

The integrals in Eqs (4.57) and (4.58) can be interpreted as cross correlation. Thus the optimum receiver for weak signals is a square law detector, followed by a correlator which finds the cross correlation between the detector output and $\big(f(t)\big)^2$, the square of the envelope of the expected signal. For the case of large signal to noise ratio, the optimum receiver is a linear detector, followed by a correlator which has for its output the cross correlation of the detector output and $f(t)$, the amplitude of the expected signal.

The distribution function for $\ell(r)$ cannot be found easily in this case. The approximation developed here will apply to the receiver designed for low signal to noise ratio, since this is the case of most interest in threshold studies. An analogous approximation for the large signal to noise ratios would be even easier to derive.

First we shall find the mean and standard deviation for the distribution of the logarithm of the likelihood ratio:
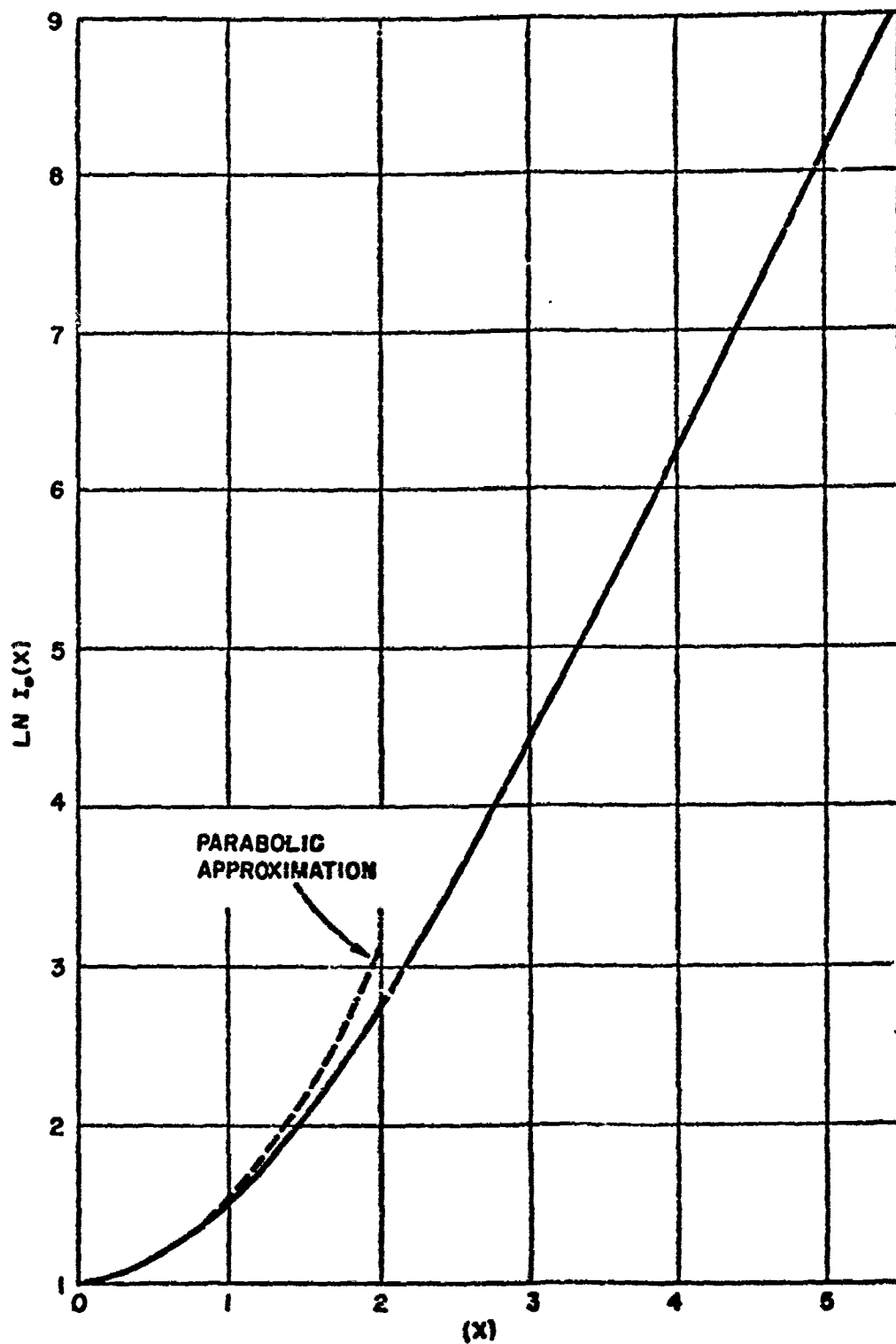
FIG. 4.9

GRAPH OF LN $I_0(X)$.

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$$\ln \ell(r) \approx -\frac{1}{2N} \sum f_i^2 + \frac{1}{4N^2} \sum_{i=1}^{n/2} r_i^2 f_i^2 \tag{4.59}$$

for the case of small signal to noise ratio. The probability density functions for each $r_i$ are

$$g_{SN}(r_i) = \frac{r_i}{N} \exp\left[-\frac{r_i^2+f_i^2}{2N}\right] I_0\left[\frac{r_i f_i}{N}\right], \quad \text{and}$$

$$g_N(r_i) = \frac{r_i}{N} \exp\left[-\frac{r_i^2}{2N}\right]. \tag{4.60}$$

The notation $g_N(r_i)$ and $g_{SN}(r_i)$ is used to distinguish these from the joint distributions of all the $r_i$ which were previously called $f_N(r)$ and $f_{SN}(r)$. The mean of each term $\frac{r_i^2 f_i^2}{N^2}$ in the sum in Eq (4.59) is

$$\mu_{SN}\left(\frac{r_i^2 f_i^2}{4N^2}\right) = \frac{f_i^2}{4N} \int_0^\infty \frac{r_i^2}{N} g_{SN}(r_i)\, dr_i$$

$$= \frac{f_i^2}{4N} \int_0^\infty \frac{r_i^3}{N^2} \exp\left[-\frac{(r_i^2+f_i^2)}{2N}\right] I_0\left(\frac{r_i f_i}{N}\right)\, dr_i$$

$$\mu_N\left(\frac{r_i^2 f_i^2}{4N^2}\right) = \frac{f_i^2}{4N} \int_0^\infty \frac{f_i^2}{N} g_N(r_i)\, dr_i = \frac{f_i^2}{4N} \int_0^\infty \frac{r_i^3}{N^2} \exp\left[-\frac{r_i^2}{2N}\right] dr_i \tag{4.61}$$

The second moment of each term $\frac{r_i^2 f_i^2}{4N^2}$ is

$$\mu_{SN}\left(\frac{r_i^4 s_i^4}{16N^4}\right) = \frac{f_i^4}{16N^2} \int_0^\infty \frac{r_i^4}{N^2} g_{SN}(r_i)\, dr_i$$

$$= \frac{f_i^4}{16N^2} \int_0^\infty \frac{r_i^5}{N^3} \exp\left[-\frac{(r_i^2+f_i^2)}{2N}\right] I_0\left(\frac{r_i f_i}{N}\right)\, dr_i$$

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$$\mu_N\left(\frac{r_1^4 r_1^4}{16N^4}\right) = \frac{r_1^4}{16N^2} \int_0^\infty \frac{r_1^4}{N^2} \; g_N(r_1) \; dr_1$$

$$= \frac{r_1^4}{16N^2} \int_0^\infty \frac{r_1^5}{N^3} \; \exp\left[-\frac{r_1^2}{2N}\right] dr_1 \tag{4.62}$$

The integrals for the case of noise alone can be evaluated easily:

$$\mu_N\left(\frac{r_1^2 r_1^2}{4N^2}\right) = \frac{r_1^2}{2N}$$

$$\mu_N\left(\frac{r_1^4 r_1^4}{16N^4}\right) = \frac{r_1^4}{2N^2}$$

The integrals for the case of signal plus noise can be evaluated in terms of the confluent hypergeometric function, which turns out for the cases above to reduce to a simple polynomial. The required formulas are collected in convenient form in the book, Threshold Signals by Lawson and Uhlenbeck.[1] The results are

$$\mu_{SN}\left(\frac{r_1^2 r_1^2}{4N^2}\right) = \frac{1}{2} \; \frac{r_1^2}{N} \left(1 + \frac{r_1^2}{2N}\right)$$

$$\mu_{SN}\left(\frac{r_1^4 r_1^4}{16N^4}\right) = \frac{1}{2} \; \frac{r_1^4}{N^2} \left(1 + \frac{r_1^2}{N} + \frac{r_1^4}{8N^2}\right) \tag{4.64}$$

Since

$$\sigma^2(z) = \mu(z^2) - \left[\mu(z)\right]^2 \quad , \tag{4.65}$$

the variance of $\dfrac{r_1^2 r_1^2}{N^2}$ is

[1] Ref. 1, p. 174

$$\sigma^2_{SN}\left(\frac{r_i^2 r_i^2}{4N^2}\right) = \frac{1}{4}\frac{r_i^4}{N^2}\left(1 + \frac{r_i^2}{N}\right)$$

$$\sigma^2_{N}\left(\frac{r_i^2 r_i^2}{4N^2}\right) = \frac{r_i^4}{4N^2} \tag{4.66}$$

For the sum of independent random variables, the mean is the sum of the means of the terms and the variance is the sum of the variances.

mean of $\ln \mathcal{L}(x)$ is

$$\mu_{SN}\left(\ln \mathcal{L}(r)\right) = -\frac{1}{2N}\sum_{i=1}^{n/2} r_i^2 + \sum_{i=1}^{n/2}\left[\frac{1}{2}\frac{r_i^2}{N} + \frac{1}{4}\frac{r_i^4}{N^2}\right] = \sum_{i=1}^{n/2}\frac{r_i^4}{4N^2}$$

$$\mu_{N}\left(\ln \mathcal{L}(r)\right) = -\sum_{i=1}^{n/2}\frac{r_i^2}{2N} + \frac{1}{2}\sum_{i=1}^{n/2}\frac{r_i^2}{N} \tag{4.67}$$

$$= 0$$

and the variance of $\ln \mathcal{L}(r)$ is

$$\sigma^2_{SN}\left(\ln \mathcal{L}(r)\right) = \sum_{i=1}^{n/2}\left(\frac{1}{4}\frac{r_i^4}{N^2} + \frac{1}{4}\frac{r_i^6}{N^3}\right)$$

$$\sigma^2_{N}\left(\ln \mathcal{L}(r)\right) = \sum_{i=1}^{n/2}\frac{r_i^4}{4N^2} \tag{4.68}$$

If the distribution functions $\ln \mathcal{L}(x)$ can be assumed to be normal, the distribution functions can be obtained immediately from the mean and standard deviation of the distribution. In some cases the normal distribution is a good approximation to the actual distribution.

Let us consider the case in which the incoming signal is a rectangular pulse which is $\frac{M}{W}$ seconds long.[1] The energy of the pulse is half its duration times the amplitude of its envelope, and therefore the amplitude has the value

$$r_i = \sqrt{\frac{2EW}{M}} \quad , \tag{4.69}$$

where $E$ is the pulse energy. It has this value on $M$ sample points and is zero at all others. For this case

$$\mu_{SN}\left(\ell n\ \mathcal{L}(r)\right) = \frac{1}{M}\frac{E^2}{N_0^2}$$

$$\mu_N\left(\ell n\ \mathcal{L}(r)\right) = 0$$

$$\sigma_{SN}^2\left(\ell n\ \mathcal{L}(r)\right) = \frac{E^2}{MN_0^2}\left(1 + \frac{2}{M}\frac{E}{N_0}\right)$$

$$\sigma_N^2\left(\ell n\ \mathcal{L}(r)\right) = \frac{E^2}{MN_0^2} \tag{4.70}$$

Also, for this case, the distribution of $\ell n\ \mathcal{L}(x)$ is approximately normal, if $M$ is much larger than one. Since it is the sum of $M$ independent random variables, all having the same distribution, it must, by the central limit theorem,[2] approach the normal distribution as $M$ becomes large. The actual distribution for the case of noise alone can be calculated in this case, since the convolution integral[3] for the $g_N(r_i)$ with itself any number of times can be

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[1] The problem of finding the distribution for the sum of $M$ independent random variables, each with a probability density function $f(x) = x\ \exp\left[-\frac{1}{2}\ (x^2+\alpha^2)\right]I_0(\alpha x)$ arises in the unpublished report by J. I. Marcum, A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix, Project Rand Report R-113. Marcum gives an exact expression for this distribution which is useful only for small values of $M$, and an approximation in Gram-Charlier series which is more accurate than the normal approximation given here. Marcum's expressions could be used in this case, and in the case presented in Section 4.6.

[2] Cramér, Ref. 14, p. 213 and 316.  [3] Cramér, Ref. 14, p. 188-9.

expressed in closed form. The density function for this distribution is plotted in Fig. 4.10 for several relatively small values of M. The distribution of $\ln l(x)$ for signal plus noise is more nearly normal than the distribution for noise alone, since the distributions $g_{SN}(r_1)$ are more nearly normal than $g_N(r_1)$.

The receiver operating characteristic for the case M = 16 is plotted in Fig. 4.11 using the normal distribution as approximation to the true distribution. In many cases it will be found that

$$\frac{1}{M} \cdot \frac{2E}{N_o} << 1 \quad . \tag{4.71}$$

In such a case the distributions have approximately the same variance. Assuming normal distribution then leads to the curves of Fig. 4.1, with

$$d = \frac{1}{4M} \left(\frac{2E}{N_o}\right)^2 \quad . \tag{4.72}$$

## 4.6 A Radar Case

This section deals with detecting a radar target at a given range. That is, we shall assume that the signal, if it occurs, consists of a train of M pulses whose time of occurrence and envelope shape are known. The carrier phase will be assumed to have a uniform distribution for each pulse independent of all others, i.e., the pulses are incoherent.

The set of signals can be described as follows:

$$s(t) = \sum_{m=0}^{M-1} f(t+m\tau) \cos(\omega t + \theta_i) \tag{4.73}$$

where the M angles $\theta_i$ have independent uniform distributions, and the function f, which is the envelope of a single pulse, has the property that
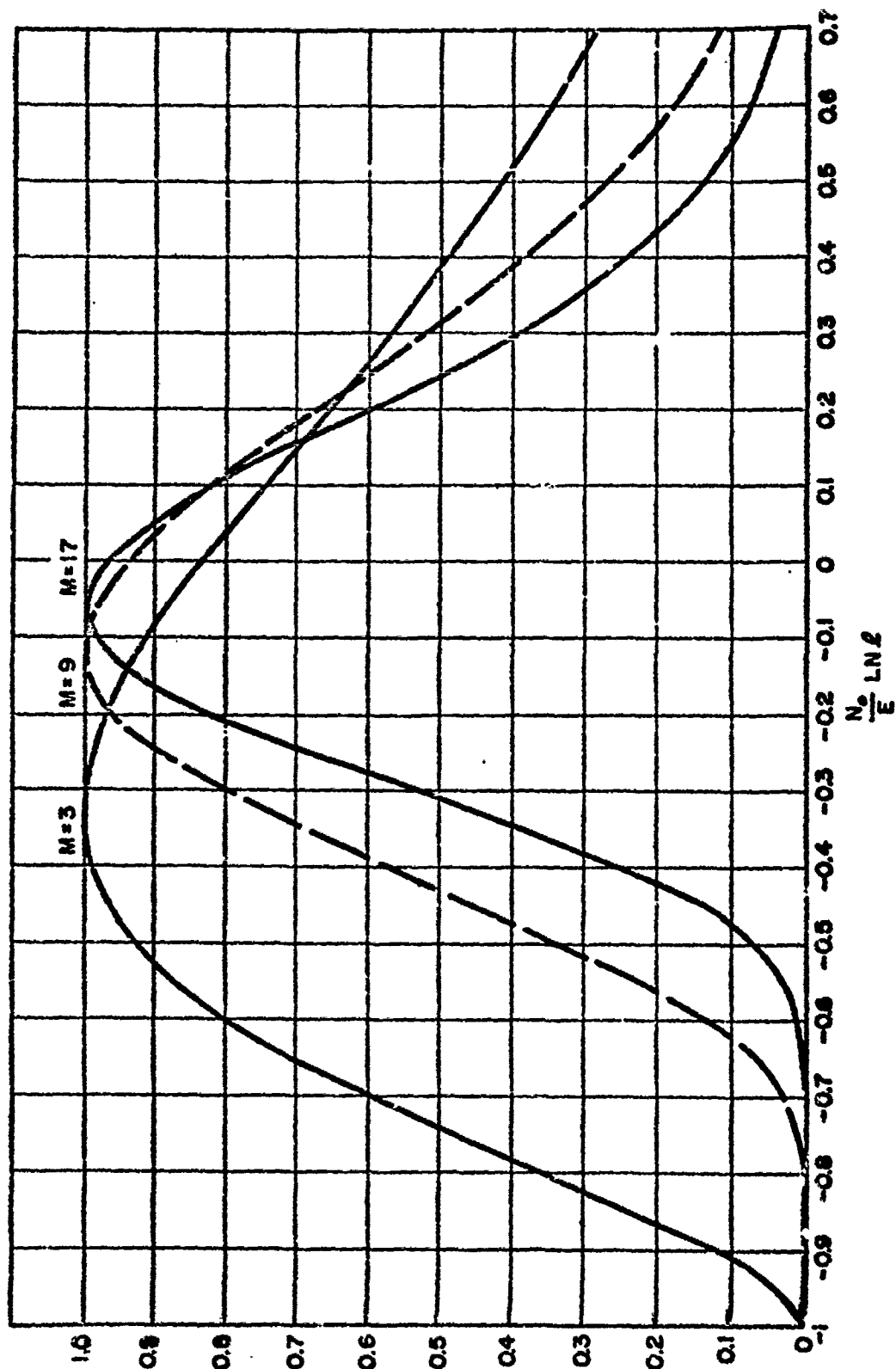
FIG. 4.10

$$\frac{PROBABLE\ DENSITY}{MAX.\ PROBABLE\ DENSITY}\ OF\ LN\ell\ VS.\ \frac{N_0}{E}\ LN\ell.$$

M=17

M=9

M=3

$\frac{N_0}{E}\ LN\ell$
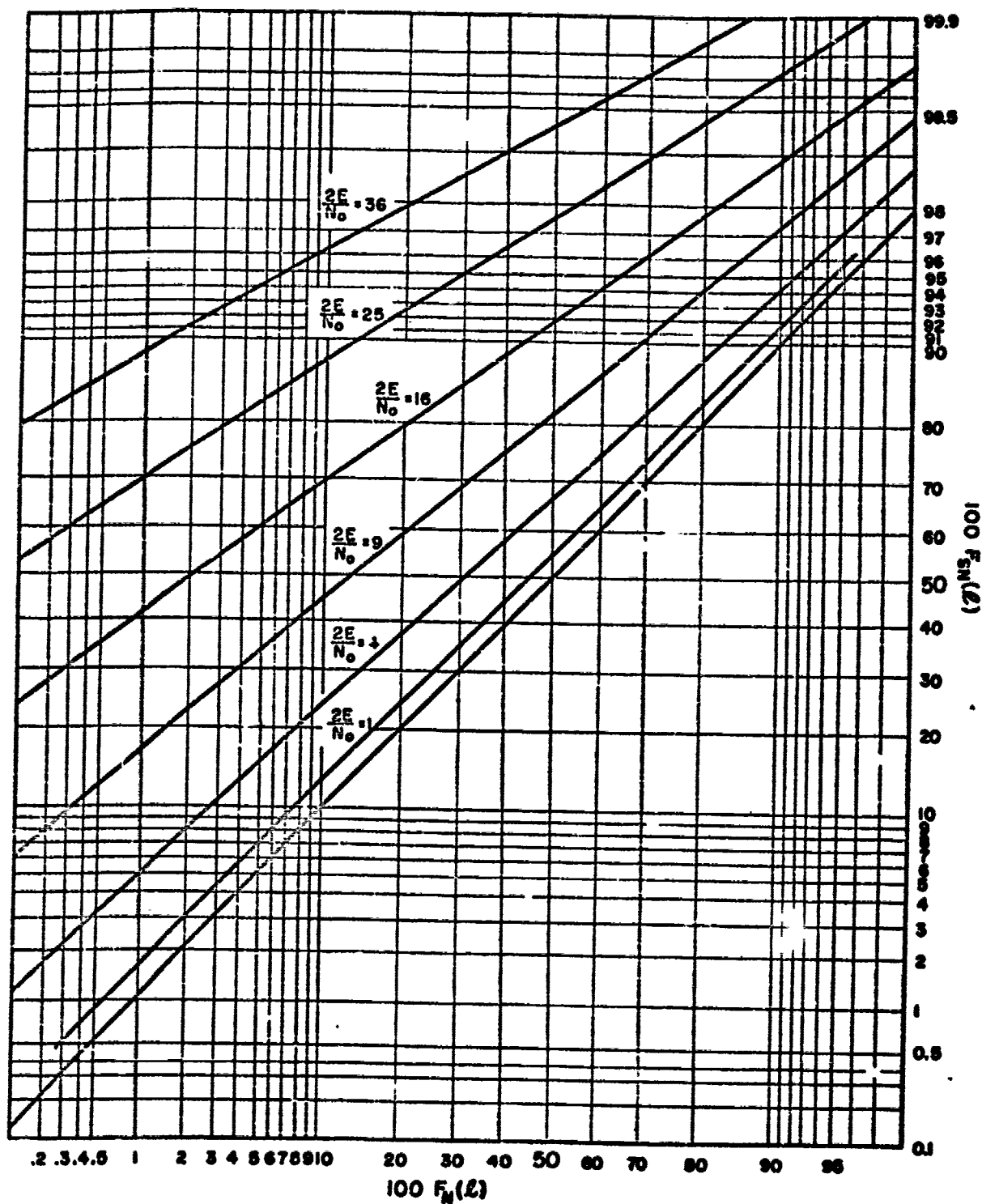
FIG. 4.11
RECEIVER OPERATING CHARACTERISTIC.

BROAD BAND RECEIVER WITH
OPTIMUM VIDEO DESIGN, M = 16.

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$$\int_0^T f(t+i\tau)\, f(t+j\tau)\, dt = \frac{2E}{M}\, \delta_{ij} \quad , \qquad (4.74)$$

where $\delta_{ij}$ is the Kronecker delta function, which is zero if $i \neq j$, and unity if $i = j$. The time $\tau$ is the interval between pulses. Eq (4.74) states that the pulses are spaced far enough so that they are orthogonal, and that the total signal energy is E.[1] The function $f(t)$ is also assumed to have no frequency components as high as $\frac{\omega}{2\pi}$ .

The likelihood ratio can be obtained by applying Eq (3.7).

$$\ell(x) = \int_R \exp\left[-\frac{E(s)}{N_0}\right] \exp\left[\frac{2}{N_0} \int_0^T s(t)\, x(t)\, dT\right]\, dP_S(s) \qquad (4.75)$$

$$= \exp\left[-\frac{E}{N_0}\right] \int_0^{2\pi} \cdots \int_0^{2\pi} \exp\left[\frac{2}{N_0} \int_0^T \sum_{m=0}^{M-1} f(t+m\tau)x(t)\cos(\omega t+\theta_m)dt\right] d\theta_0 \cdots d\theta_{M-1} \qquad (4.76)$$

The integral can be evaluated, as in Section 4.3, and

$$\ell(x) = \exp\left[-\frac{E}{N_0}\right] \prod_{m=0}^{M-1} I_0\left(\frac{r_m}{N}\right) \quad , \qquad (4.77)$$

where

$$\left(\frac{r_m}{N}\right)^2 = \left[\frac{2}{N_0} \int_0^T f(t+m\tau)x(t)\cos\omega t\,dt\right]^2 + \left[\frac{2}{N_0} \int_0^T f(t+m\tau)\, x(t)\sin\omega t\,dt\right]^2 \qquad (4.78)$$

This quantity $r_m$ is almost identical with the quantity $r$ which appeared in the discussion of the case of the signal known except for carrier phase, Section 4.3. In fact, each $r_m$ could be obtained in a receiver in the manner

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[1] The factor 2 appears in (4.74) because $f(t)$ is the pulse envelope; the factor M appears because the total energy E is M times the energy of a single pulse.

described in that section. The $r_o$ is connected with the first pulse; it could be obtained by designing a ideal filter for the signal

$$s_o(t) = f(t) \cos (\omega t + \theta) \qquad (4.79)$$

for any value of the phase angle $\theta$, and putting the output through a linear detector. The output will be $\frac{N_o}{2} \frac{r_o}{N}$ at some instant of time $t_o$ which is determined by the time delay of the filter. The other quantities $r_m$ differ only in that they are associated with the pulses which come later. The output of the filter at time $t_o + m\tau$ will be $\frac{N_o}{2} \frac{r_m}{N}$ .

It is convenient to have the receiver calculate the logarithm of the likelihood ratio,

$$\ln \ell(x) = -\frac{E}{N_o} + \sum_{m=0}^{M-1} \ln I_o \left( \frac{r_m}{N} \right) \qquad (4.80)$$

Thus the $\ln I_o \left( \frac{r_m}{N} \right)$ must be found for each $r_m$, and these M quantities must be added. As in the previous section, $\frac{r_m}{N}$ will usually be small enough so that $\ln I_o(x)$ can be approximated by $\frac{x^2}{4}$ .[1] The quantities $\frac{1}{4} \left( \frac{r_m}{N} \right)^2$ can be found by using a square law detector rather than a linear detector, and the outputs of the square law detector at times $t_o$, $t_o + \tau$, ..., $t_o + (M-1)\tau$ then must be added. The ideal system thus consists of an i.f. amplifier with its passband matched to a single pulse,[2] a square law detector (for the threshold signal case), and an integrating device.

We shall find normal approximations for the distribution functions of the logarithm of the likelihood ratio using the approximation

$$\ln I_o \left( \frac{r_m}{N} \right) \approx \frac{r_m^2}{4N^2} \qquad (4.81)$$

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[1] See Fig. 4.9.

[2] It is usually most convenient to make the ideal filter (or an approximation to it) a part of the i.f. amplifier.

which is valid for small values of $\frac{r_m}{N}$, [1]

$$\ln \ell \approx -\frac{E}{N_0} + \sum_{n=0}^{M-1} \frac{1}{4} \left(\frac{r_m}{N}\right)^2 . \qquad (4.82)$$

The distributions for the quantities $r_m$ are independent; this follows from the fact that the individual pulse functions $f(t+m\tau) \cos(\omega t + \theta_m)$ are orthogonal. The distribution for each is the same as the distribution for the quantity $r$ which appears in the discussion of the signal known except for phase; the same analysis applies to both cases. Thus, by Eq $(4.22)$[2]

$$P_N\left(\frac{r_m}{N}\sqrt{\frac{N_0 M}{2E}} \geq \alpha\right) = \exp\left[-\frac{\alpha^2}{2}\right]$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.83)$

$$P_N\left(\frac{r_m}{N} \geq a\right) = \exp\left[-\frac{a^2 N_0 M}{2E}\right] ,$$

and by $(4.26)$,

$$P_{SN}\left(\sqrt{\frac{N_0 M}{2E}}\frac{r_m}{N} \geq \alpha\right) = \exp\left[-\frac{E}{N_0}\right]\int_{\alpha}^{\infty} \alpha \exp\left[-\frac{\alpha^2}{2}\right] I_0\left(\alpha\sqrt{\frac{2E}{N_0 M}}\right) d\alpha$$

or

$$P_{SN}\left(\frac{r_m}{N} \geq a\right) = \frac{N_0 M}{2E}\exp\left[-\frac{E}{N_0 M}\right]\int_{a}^{\infty} a \exp\left(-\frac{a^2 N_0 M}{4E}\right) I_0(a)\, da \qquad (4.84)$$

The density functions can be obtained by differentiating $(4.83)$ and $(4.84)$:

$$\sigma_N\left(\frac{r_m}{N}\right) = \frac{M N_0}{2E}\left(\frac{r_m}{N}\right)\exp\left[-\left(\frac{r_m}{N}\right)^2\left(\frac{N_0 M}{4E}\right)\right] ,$$

$$\sigma_{SN}\left(\frac{r_m}{N}\right) = \frac{M N_0}{2E}\left(\frac{r_m}{N}\right)\exp\left[-\frac{E}{M N_0}\right]\exp\left[-\left(\frac{r_m}{N}\right)^2\left(\frac{N_0 M}{4E}\right)\right] I_0\left(\frac{r_m}{N}\right). \qquad (4.85)$$

---

[1] See footnote 1, p. 37.

[2] The M appears in the following equations because the energy of a single pulse is $\frac{E}{M}$ rather than E.

This is the same situation, mathematically, as appeared in the previous section on page 34. The standard deviation and the mean for the logarithm of the likelihood ratio can be found in the same manner, and they are

$$\mu_{SN} (\ln \mathcal{L}) = \frac{E^2}{MN_o^2}$$

$$\mu_N (\ln \mathcal{L}) = 0$$

$$\sigma_{SN}^2 (\ln \mathcal{L}) = \frac{E^2}{MN_o^2} \left(1 + \frac{2E}{MN_o}\right)$$

$$\sigma_N^2 (\ln \mathcal{L}) = \frac{E^2}{MN_o^2} \tag{4.86}$$

If the distributions can be assumed normal, they are completely determined by their means and variances. These formulas are identical with the formulas (4.70) on page 37 of the previous section. The problem is the same, mathematically, and the discussion and receiver operating characteristic curves at the end of Section 4.6 apply to both cases.

## 4.7 Approximate Evaluation of an Optimum Receiver

In order to obtain approximate results for the remaining two cases, the assumption is made that in these cases the receiver operating characteristic can be approximated by the curves of Fig. 4.1, i.e., that the logarithm of the likelihood ratio is approximately normal. This section discusses the approximation and a method for fitting the receiver operating characteristic to the curves of Fig. 4.1.

It was pointed out in Section 2.5.1 of Part I of this report that $F_{SN}(\mathcal{L})$ can be calculated if $F_N(\mathcal{L})$ is known. It was further pointed out that the $n^{th}$ moment of the distribution $F_N(\mathcal{L})$ is the $(n-1)^{th}$ moment of the distribution $F_{SN}(\mathcal{L})$. Hence, the mean of the likelihood ratio with noise alone is

unity, and if the variance of the likelihood ratio with noise alone is $\sigma_N^2$, the second moment with noise alone, and hence the mean with signal plus noise is $1 + \sigma_N^2$. Thus the difference between the means, and the variance with noise alone are the same number $\sigma_N^2$. This number probably characterizes the receiver reliability better than any other single number.

Suppose the logarithm of the likelihood ratio has a normal distribution with noise alone, i.e.,

$$F_N(\ell) = \frac{1}{\sqrt{2\pi d}} \int_{\ell n \, \ell}^{\infty} \exp\left[-\frac{(x-m)^2}{2d}\right] dx, \qquad (4.87)$$

where m is the mean and d the variance of the logarithm of the likelihood ratio. The $n^{th}$ moment of the likelihood ratio can be found as follows:

$$\mu_N(\ell^n) = \int_0^{\infty} \ell^n \, dF_n(\ell) = \frac{1}{\sqrt{2\pi d}} \int_{-\infty}^{\infty} \exp\left[nx\right] \exp\left[-\frac{(x-m)^2}{2d}\right] dx \quad, \qquad (4.88)$$

where the substitution $\ell = \exp x$ has been made. The integral can be evaluated by completing the square in the exponent and using the fact that

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2d}\right] dx = \sqrt{2\pi d} \qquad,$$

$$\mu_N(\ell^n) = \exp\left[\frac{n^2 d}{2} + mn\right] \quad. \qquad (4.89)$$

In particular, the mean of $\ell(x)$, which must be unity, is

$$\mu_N(\ell) = 1 = \exp\left[\frac{d}{2} + m\right] \quad, \qquad (4.90)$$

and therefore

$$m = -\frac{d}{2} \qquad, \qquad (4.91)$$

The variance of $\ell(x)$ with noise alone is $\sigma_N^2$, and therefore the second moment

of $\ell(x)$ is

$$\mu_N(\ell^2) = \left[\mu_N(\ell)\right]^2 + \sigma_N^2(\ell) = 1 + \sigma_N^2(\ell) \qquad , \quad (4.92)$$

and this must agree with (4.89).

$$\mu_N(\ell^2) = 1 + \sigma_N^2 = \exp\left[2d + 2m\right] = \exp\left[d\right] \qquad (4.93)$$

and therefore

$$d = \ell n \left(1 + \sigma_N^2\right) \qquad . \qquad (4.94)$$

The distribution of likelihood ratio with signal plus noise can be found by applying Theorem 8.[1]

$$dF_{SN}(\ell) = \ell \, dF_N(\ell) \qquad ,$$

$$F_{SN}(\ell) = - \int_\ell^\infty dF_N(\ell) \qquad . \qquad (4.95)$$

Substituting for $F_N(\ell)$ from (4.87), and letting $\ell = \exp x$ yields

$$F_{SN}(\ell) = \frac{1}{\sqrt{2\pi d}} \int_{\ell n \ell}^\infty \exp[x] \exp\left[-\frac{\left(x + \frac{d}{2}\right)^2}{2d}\right] dx$$

$$= \frac{1}{\sqrt{2\pi d}} \int_{\ell n \ell}^\infty \exp\left[-\frac{\left(x - \frac{d}{2}\right)^2}{2d}\right] dx \qquad . \qquad (4.96)$$

Thus the distribution of $\ell n \, \ell$ is normal also when there is signal plus noise, in this case with mean $\frac{d}{2}$ and variance d.

In summary, the variance $\sigma_N^2$ of the likelihood ratio probably measures the receiver reliability better than any other single number. If the logarithm of the likelihood ratio has a normal distribution, then this distribution, and

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[1] See Part I, Section 2.4.

hence the signal plus noise distribution, are completely determined if $\sigma_N^2$ is given. Both distributions of $\ln \ell(x)$ are normal with the same variance d, and the difference of the means is d. The receiver operating characteristic curves are those plotted in Fig. 4.1, with the parameter d related to $\sigma_N^2$ by the equation

$$d = \ln (1 + \sigma_N^2) \qquad . \qquad (4.94)$$

In the case of a signal known exactly, this is the distribution which occurs. In the cases of Section 4.4, Section 4.5, and Section 4.6 this distribution is found to be the limiting distribution when the number of sample points is large. Certainly in most cases the distribution has this general form. Thus it seems reasonable that useful approximate results could be obtained by calculating only $\sigma_N^2$ for a given case and assuming that the receiver reliability is approximately the same as if the logarithm of the likelihood ratio had a normal distribution. On this basis, $\sigma_N^2(\ell)$ is calculated in the following sections for two cases, and the assertion is made that the receiver reliability is given approximately by the receiver operating characteristic curves of Fig. 4.1 with $d = \ln (1 + \sigma_N^2)$.

## 4.8  Signal Which is One of M Orthogonal Signals

The following case has several applications, which will be discussed in Section 5.3. The importance of this case, and the one which follows it, lies in the fact that the uncertainty of the signal distribution can be varied by changing the parameter M.

Suppose that the set of expected signals includes just M orthogonal functions $s_k(t)$, all of which have the same probability, the same energy E, and

are orthogonal. That is,

$$\int_0^T s_k(t)\, s_q(t)\, dt \;=\; E\,\delta_{kq} \tag{4.97}$$

Then the likelihood ratio can be found from Eq (3.7) to be

$$\ell(x) \;=\; \sum_{k=1}^{M} \frac{1}{M} \exp\left[-\frac{E}{N_0}\right] \exp\left[\frac{1}{N}\sum_{i=1}^{n} x_i s_{ki}\right]$$

$$=\; \frac{1}{M}\sum_{k=1}^{M} \exp\left[\frac{1}{N}\sum_{i=1}^{n} x_i s_{ki} - \frac{E}{N_0}\right] \tag{4.98}$$

where $s_{ki}$ are the sample values of the function $s_k(t)$.

It should be clear that with noise alone, the terms $\frac{1}{N}\sum_{i=1}^{n} x_i s_{ki}$

have a Gaussian distribution with mean zero and variance $\sum_{i=1}^{n} \frac{s_{ki}^2}{N} = \frac{2E}{N_0}$. [1]

Furthermore, the M different quantities $\frac{1}{N}\sum_{i=1}^{n} x_i s_{ki}$ are independent, since the

functions $s_k(t)$ are orthogonal. It follows that the terms $\exp\left[\frac{1}{N}\sum_{i=1}^{M} x_i s_{ki} - \frac{E}{N_0}\right]$

are independent.

Since the logarithm of each term $Z = \exp\left[\frac{1}{N}\sum_{i=1}^{n} x_i s_{ki} - \frac{E}{N_0}\right]$ has a

normal distribution with mean $-\frac{E}{N_0}$ and variance $\frac{2E}{N_0}$, the moments of the distri-

bution can be found from Eq (4.89). The $n^{th}$ moment is

$$\mu_N(Z^n) \;=\; \exp\left[n(n-1)\frac{E}{N_0}\right] \quad . \tag{4.99}$$

----

[1] The reasoning is the same as that on page 9.

It follows that the mean of each term is unity, and the variance is

$$\sigma_N^2(Z) = \mu(Z^2) - \left[\mu(Z)^2\right] = \exp\left[\frac{2E}{N_0}\right] - 1 \quad . \tag{4.100}$$

The variance of a sum of independent random variables is the sum of the variances of the terms. Therefore

$$\sigma_N^2(M\ell) = M\left[\exp\left(\frac{2E}{N_0}\right) - 1\right] \quad , \tag{4.101}$$

and it follows that the variance of the likelihood ratio is

$$\sigma_N^2(\ell) = \frac{1}{M}\left[\exp\left(\frac{2E}{N_0}\right) - 1\right] \quad . \tag{4.102}$$

It was pointed out in Section 4.7, page 47 that the receiver operating characteristic curves are approximately those of Figure 4.1, with

$$d = \ell n \,(1 + \sigma_N^2) = \ell n\left(1 - \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2E}{N_0}\right)\right) \tag{4.103}$$

This equation can be solved for $\frac{2E}{N_0}$ :

$$\frac{2E}{N_0} = \ell n\left[1 + M\,(e^d - 1)\right] \quad . \tag{4.104}$$

Curves of $\frac{2E}{N_0}$ for constant d are plotted in Fig. 4.12. They show how much the signal energy must be increased when the number of possible signals increases.

## 4.9  Signal Which is One of M Orthogonal Signals with Unknown Carrier Phase

Consider the case in which the set of expected signals includes just M different amplitude modulated signals which are known except for carrier phase. Denote the signals by

$$e_k(t) = f_k(t) \cos\,(\omega t + \theta) \quad . \tag{4.105}$$
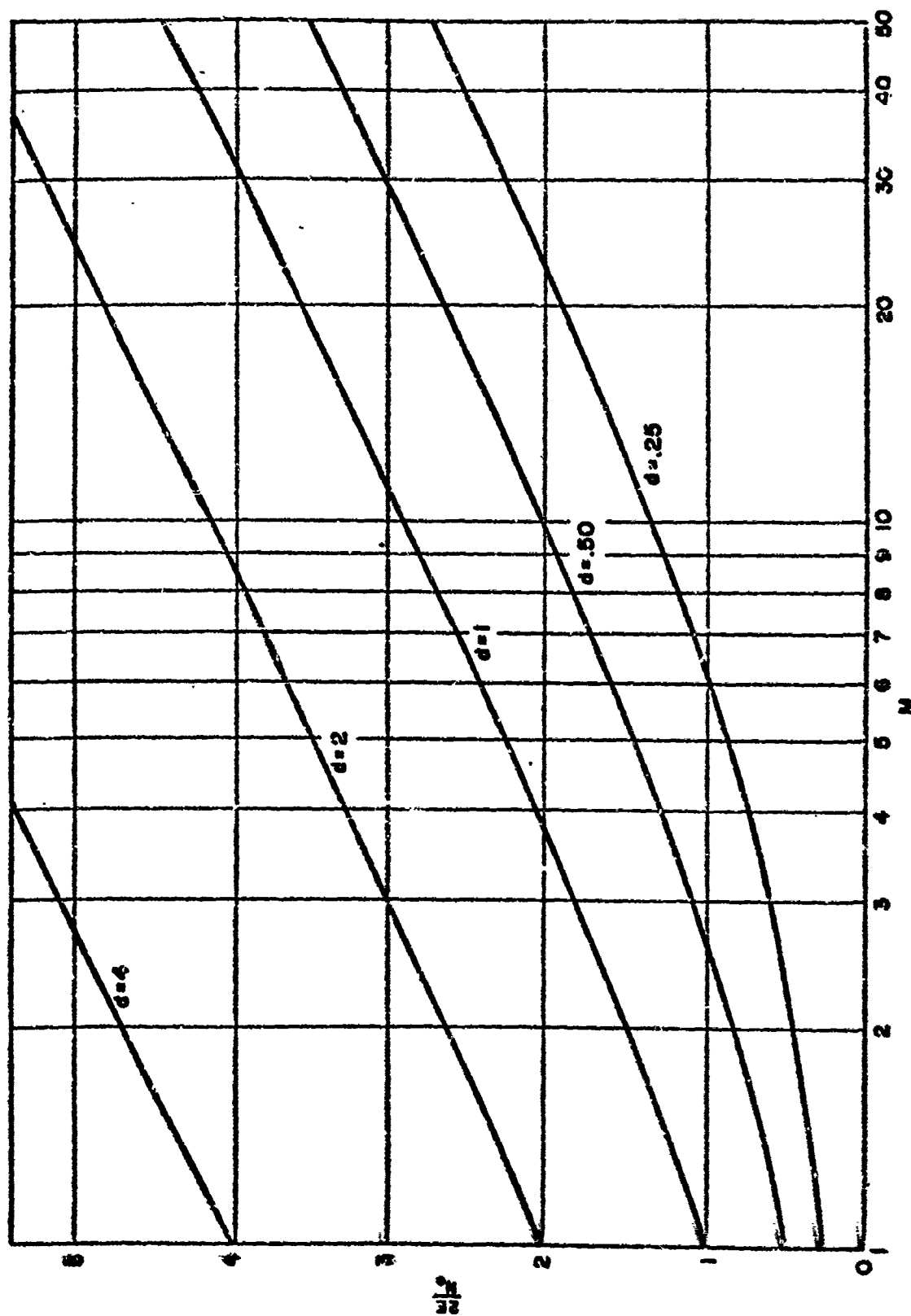
FIG. 4.12. SIGNAL ENERGY AS A FUNCTION OF M AND d.
SIGNAL ONE OF M ORTHOGONAL SIGNALS.

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It will be assumed further that the functions $f_k(t)$ all have the same energy E and are orthogonal, i.e.,

$$\int_0^T f_k(t)\, f_q(t)\, dt \;=\; 2E\, \delta_{kq} \quad , \qquad (4.106)$$

where the 2 is introduced because the f's are the signal amplitudes, not the actual signal functions. Also, let the $f_k(t)$ be band-limited to contain no frequencies as high as $\omega$. Then it follows that any two signal functions with different envelope functions will be orthogonal. Let us assume also that the distribution of phase $\theta$ is uniform, and that the probability for each envelope function is $\frac{1}{N}$.

With these assumptions, the likelihood ratio can be obtained from Eq (3.7), and it is

$$\ell(x) \;=\; \frac{1}{N} \sum_{k=1}^{M} \frac{1}{2\pi} \int_0^{2\pi} \exp\left[ \frac{1}{N} \sum_{i=1}^{n} x_i s_{ki} - \frac{E}{N_o} \right] d\theta \qquad (4.107)$$

where $s_{ki}$ are the sample values of $s_k(t)$, and hence depend upon the phase $\theta$. The integration is the same as in the case of the signal known except for phase, and the result can be obtained from Eq (4.19)

$$\ell(x) \;=\; \frac{1}{N} \sum_{k=1}^{M} \exp\left[ -\frac{E}{N_o} \right] I_0\left( \frac{r_k}{N} \right) \quad , \qquad (4.108)$$

where

$$r_k \;=\; \sqrt{\left( \sum_i x_i f_k(t_i) \cos \omega t_i \right)^2 + \left( \sum_i x_i f_k(t_i) \sin \omega t_i \right)^2} \quad . \qquad (4.109)$$

Now the problem is to find $\sigma_N^2(\ell)$. The variance of each term in the sum in Eq (4.108) can be found, since the distribution function with noise alone can be found as in Section 4.3. Since the $f_k(t)$ are orthogonal, the

distributions of the $r_k$ are independent, and the terms in the sum in Eq (4.107) are independent. Then the variance of the likelihood ratio, $\sigma_N^2(\ell)$ is the sum of the variances of the terms, divided by $M^2$. [1]

The distribution function for each term $\exp\left[-\dfrac{E}{N_0}\right] I_0\left(\dfrac{r_k}{M}\right)$ is given in Section 4.3 by Eq (4.21) and (4.22). If $\alpha$ is defined by the equation

$$\beta = \exp\left[-\frac{E}{N_0}\right] I_0\left(\alpha\sqrt{\frac{2E}{N_0}}\right) , \qquad (4.10)$$

then the distribution function in the presence of noise for each term in Eq (4.108) is

$$F_N^{(k)}(\beta) = \exp\left[-\frac{\alpha^2}{2}\right] . \qquad (4.111)$$

The mean value of each term is

$$\mu^{(k)}(\beta) = \int_0^\infty \beta dF_N^{(k)}(\beta) = \int_0^\infty \exp\left[-\frac{E}{N_0}\right] I_0\left(\sqrt{\frac{2E}{N_0}}\,\alpha\right)\alpha \exp\left[-\frac{\alpha^2}{2}\right]d\alpha . \qquad (4.112)$$

This can be evaluated,[2] and the result is that $\mu^{(k)}(\beta) = 1$.

The second moment of each term is

$$\mu_N^{(k)}(\beta^2) = \int_0^\infty \beta^2 \, dF_N^{(k)}(\beta)$$

$$= \int_0^\infty \exp\left[-\frac{2E}{N_0}\right]\left[I_0\left(\alpha\sqrt{\frac{2E}{N_0}}\right)\right]^2 \alpha \exp\left[-\frac{\alpha^2}{2}\right]d\alpha \qquad (4.113)$$

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[1] Kramer, Ref. 14, p. 168.

[2] Lawson and Uhlenbeck, Ref. 1, p. 174.

The integral is evaluated in Appendix E, and the result is

$$\mu_M^{(k)}(\beta^2) = I_o\left(\frac{2E}{N_o}\right) \quad . \tag{4.114}$$

The variance is

$$\left[\sigma_M^{(k)}(\beta)\right]^2 = \mu^{(k)}(\beta^2) - \left[\mu^{(k)}(\beta)\right]^2 = I_o\left(\frac{2E}{N_o}\right) - 1 \quad . \tag{4.115}$$

It follows that the variance of $M\ell$ is

$$\sigma_M^2(M\ell) = M\left[I_o\left(\frac{2E}{N_o}\right) - 1\right] \quad , \text{ and} \tag{4.116}$$

$$\sigma_M^2(\ell) = \frac{1}{M}\left[I_o\left(\frac{2E}{N_o}\right) - 1\right] \quad , \tag{4.117}$$

since the variance for the sum of independent random variables is the sum of the variances.

If the approximation described in Section 4.7 is used, the receiver operating characteristic curves are approximately those of Fig. 4.1, with

$$d = \ell n\,(1 + \sigma_M^2) = \ell n\left(1 - \frac{1}{M} + \frac{1}{M}\,I_o\left(\frac{2E}{N_o}\right)\right). \tag{4.118}$$

Curves of $\frac{2E}{N_o}$ vs $M$ for constant $d$ are plotted in Fig. 4.13.
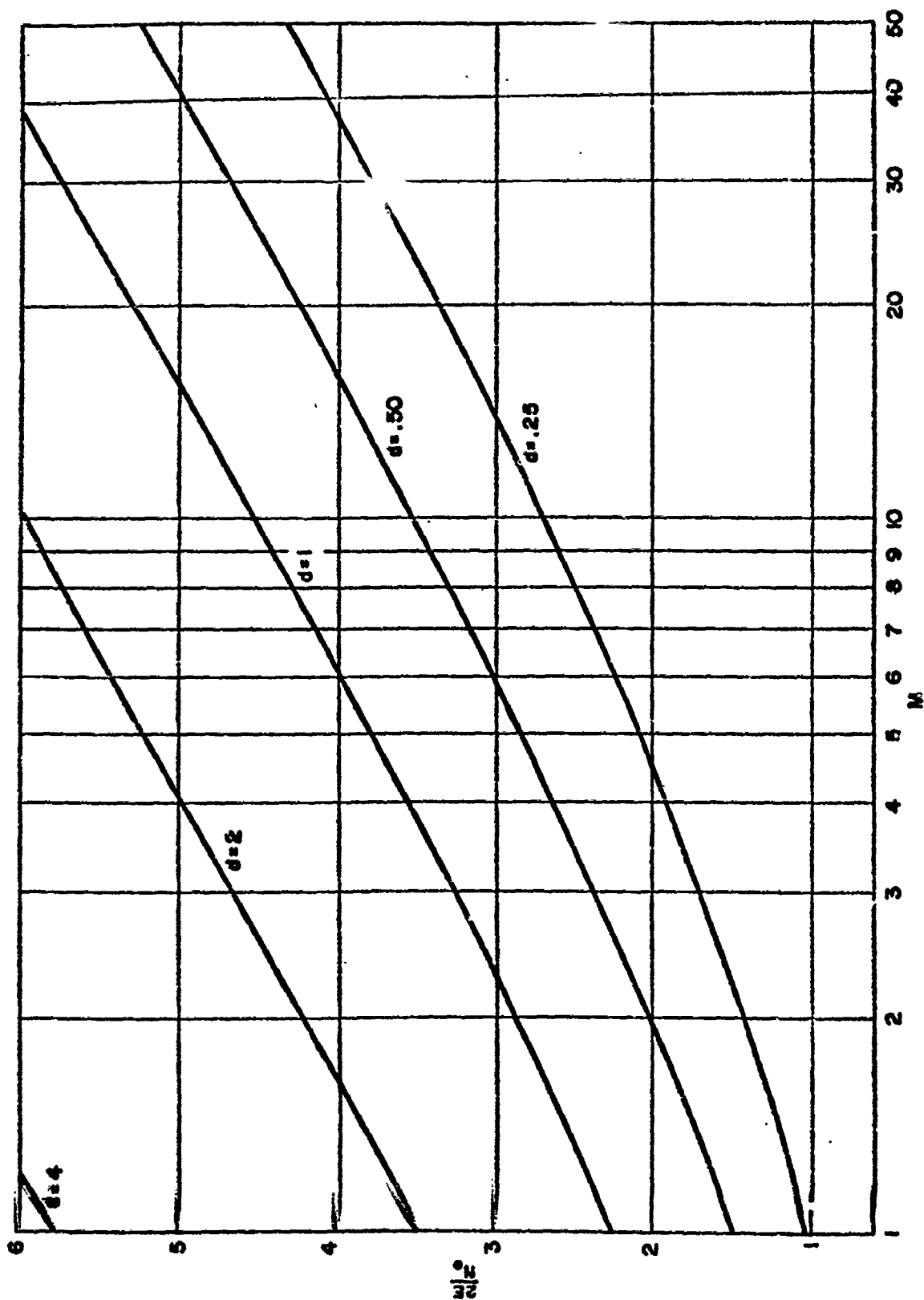
FIG. 4.13. SIGNAL ENERGY AS A FUNCTION OF M AND d.
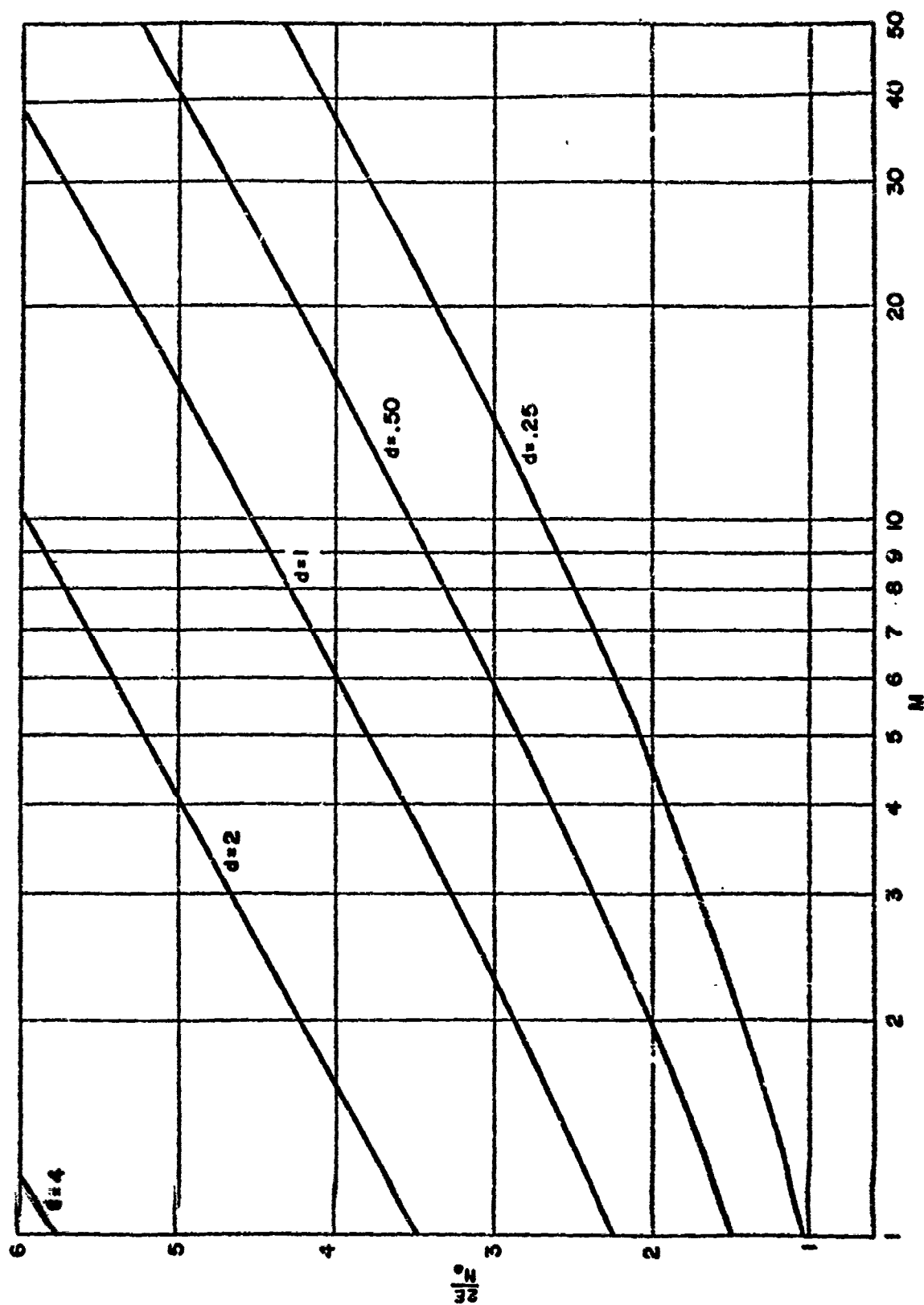SIGNAL ONE OF M ORTHOGONAL SIGNALS
KNOWN EXCEPT FOR PHASE

FIG. 4.13. SIGNAL ENERGY AS A FUNCTION OF M AND d.
SIGNAL ONE OF 4 ORTHOGONAL SIGNALS
KNOWN EXCEPT FOR PHASE

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## 5. DISCUSSION OF THE SPECIAL CASES

### 5.1 Receiver Evaluation

5.1.1 Introduction. In Section 2.5 it was shown that the receiver reliability can be determined from the distribution functions for likelihood ratio. In particular an optimum criterion receiver operating at the level $\beta$ of likelihood ratio has false alarm probability $P_N(A) = F_N(\beta)$, and probability of detection $P_{SN}(A) = F_{SN}(\beta)$. The functions $F_N(\beta)$ and $F_{SN}(\beta)$ are calculated in Section 4 for a number of special cases.

For the purpose of discussing receiver reliability it is sufficient to have the receiver operating characteristic in which $F_{SN}(\beta)$ is plotted as a function of $F_N(\beta)$. In this discussion $\beta$ plays only a secondary role.

The receiver operating characteristic shown in Figure 5.1 applies to several cases. Among them is the case of the signal known exactly, with the parameter d equal to $\frac{2E}{N_0}$ , twice the ratio of signal energy to noise power per unit bandwidth.[1] Thus, for example, if the signal is a voltage which is a known function of time, and if the signal energy is twice the noise power per unit bandwidth, theoretically a receiver could be built with false alarm probability of 0.25 and a probability of detection 0.90. If the false alarm probability is required to be no greater than 0.10, the probability of detection can be made no greater than 0.76. If the false alarm probability is required to be no greater than 0.023 and the probability of detection is to be at least 0.98, the signal energy must be at least eight times the noise power per unit bandwidth.

5.1.2 Comparison of the Simple Cases. Several curves for the case of a signal known except for phase are shown in Fig. 5.2 for some of the same values
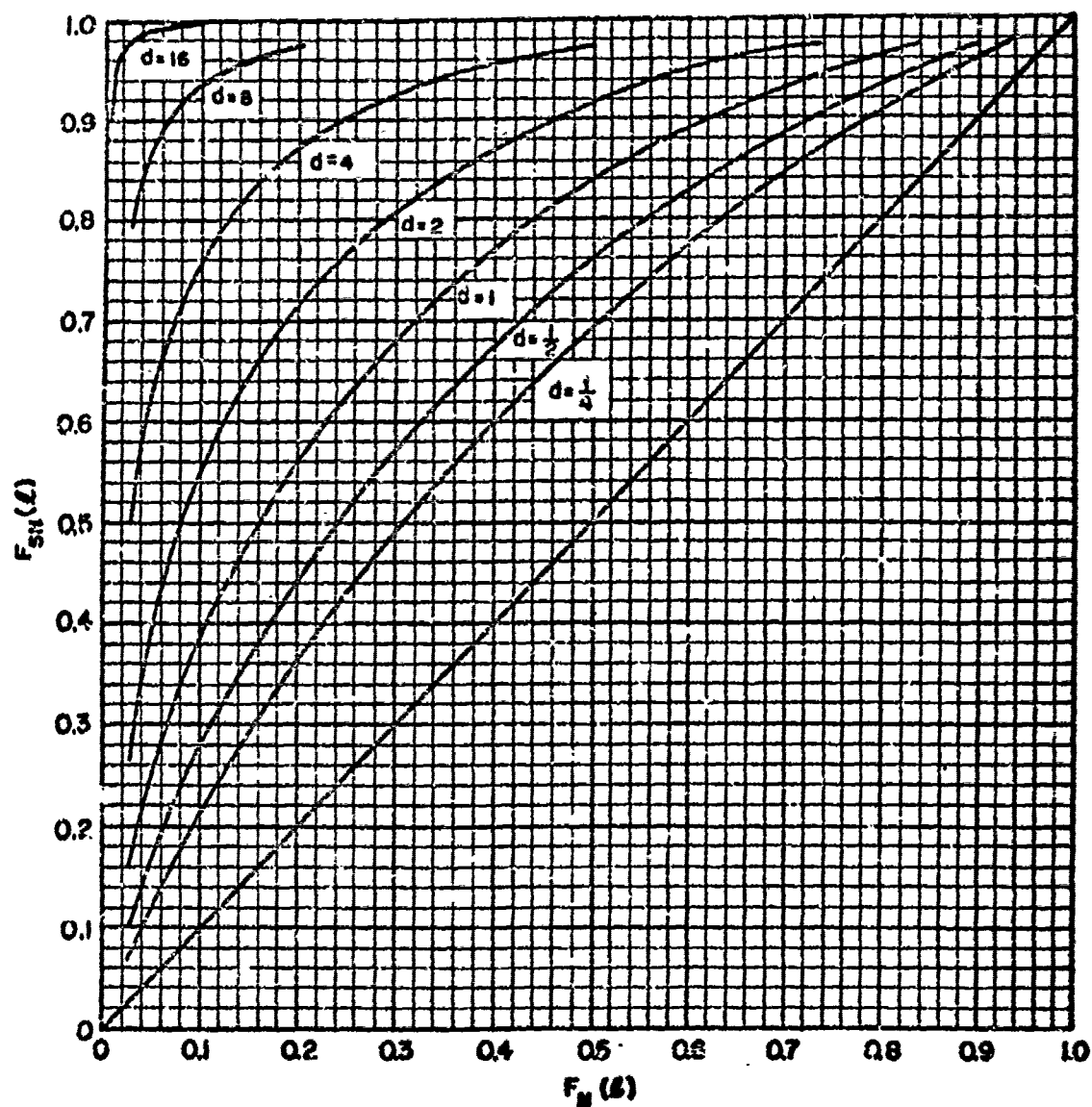
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[1]See Section 4.2.

FIG. 5.1

RECEIVER OPERATING CHARACTERISTIC.
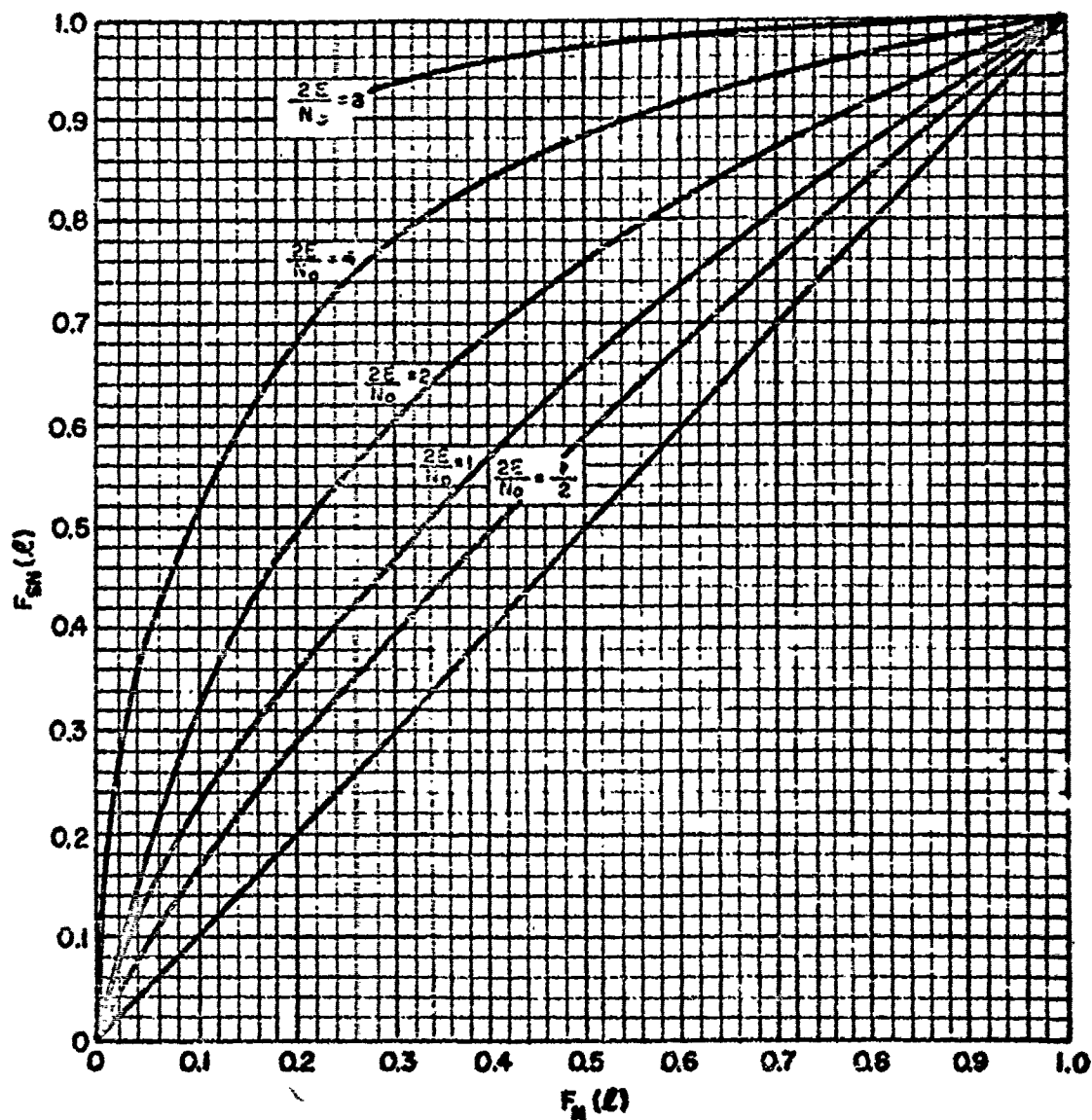
$\ln L$ IS A NORMAL DEVIATE WITH $\sigma_N^2 = \sigma_{SN}^2$, $(M_{SN} - M_N)^2 = d \, \sigma_N^2$.

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FIG. 5.2

RECEIVER OPERATING CHARACTERISTIC.

SIGNAL KNOWN EXCEPT FOR PHASE.

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of the ratio $\frac{2E}{N_0}$ as appear in Fig. 5.1. The curves for a given energy lie below those for the case of the signal known exactly; with a given false alarm probability and a given value of $\frac{2E}{N_0}$, one cannot achieve as high a probability of detection if the carrier phase of the signal is unknown.

It was found that in several cases the distribution of $\ln \mathcal{L}(x)$ approached a normal distribution as a limiting case, and that in the limit the variance with signal plus noise and the variance with noise alone are equal. In any such case the curves of Fig. 5.1 apply, and a comparison of those cases is simplified. For example, in the case of a signal which is a sample of white Gaussian noise, it was found that if the number of sample points is large and the signal to noise ratio is small, then this approximation applies, with

$$d = (2n-1)\left(1 - \sqrt{\frac{N}{N+S}}\right)^2 \approx \frac{n}{2}\left(\frac{S}{N}\right)^2 \quad . \tag{4.37}$$

Other curves for this case, some with small sample number and moderate signal to noise ratio, are given in Figs. 4.6 and 4.7. The exact equations for the distribution are Eqs (4.33) and (4.34).

The following two cases lead to the same receiver operating characteristic in the approximation considered in Sections 4.5 and 4.6: (1) the broad band receiver with optimum video design, with a pulse signal, and (2) the optimum receiver for a train of pulses with incoherent phase. In the first case the parameter M was taken as the product of the total bandwidth of the receiver and the pulse width of the signal. In the case of the train of pulses, M is the number of pulses. In each case E is the total energy of the signals. Approximate receiver operating characteristics are plotted in Fig. 4.10. Small signal to noise ratio and large M lead to the distributions for which Fig. 5.1 is

plotted, this time with

$$d = \frac{1}{4M}\left(\frac{2E}{N_0}\right)^2 \quad . \tag{4.71}$$

### 5.1.3 An Approximate Evaluation of Optimum Receivers.

Some simpler evaluation of receivers was needed because of the difficulty in solving directly for the distribution function of likelihood ratio in any cases more complicated than the ones already mentioned. It seemed reasonable to approximate the actual receiver operating characteristic by the curves given in Fig. 5.1, finding in some manner the value of the detection index d which leads to the best fit of the approximate curve to the real curve. This is suggested by the occurrence of the curves of Fig. 5.1 in four of the five cases already discussed. Also, any receiver operating characteristic must have in common with the curves of Fig. 5.1 that its slope is positive and its second derivative is negative, and that it must start at the lower left hand corner and end at the upper right hand corner of the graph.

It is shown in Section 2.5.2 that the variance $\sigma_N^2$ of the likelihood ratio when there is noise alone is the same as the difference of the means of likelihood ratio with noise alone and with signal plus noise. This parameter $\sigma_N^2$ seems to characterize signal detectability better than any other single number. In Section 4.7, it is shown that if $\sigma_N^2$ is given and the logarithm of the likelihood ratio is assumed to have a normal distribution with noise alone, then it follows that the logarithm of the likelihood ratio with signal plus noise also has a normal distribution with the same variance, and thus the receiver operating characteristic is that of Fig. 5.1. The index d is given by

$$d = \ln\left(1 + \sigma_N^2\right) \quad . \tag{4.94}$$

It seems reasonable that the curves be fitted on this basis, i.e., that $\sigma_\Pi^2$ be determined for the actual situation and the approximate receiver operating characteristic graph be taken as the curve of Fig. 5.1 with index d given by the above Eq (4.94).

    5.1.4 The Signal One of M Orthogonal Signals. The methods of the previous section have been applied to the case where the operator knows that the signal, if it occurs, will be one of M orthogonal functions of equal energy. Orthogonal, of course, means that the functions have zero cross correlation, i.e., $f(t)$ and $g(t)$ are orthogonal if

$$\int_0^T f(t)\, g(t)\, dt = 0 \tag{5.1}$$

where the integration is over the observation interval. The value obtained for $\sigma_M^2$ is[1]

$$\sigma_M^2 = \frac{1}{M}\left[\exp\left(\frac{2E}{N_0}\right) - 1\right] \tag{4.102}$$

and so the approximate receiver operating characteristic is that of Fig. 5.1 with

$$d = \ell n\left[1 - \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2E}{N_0}\right)\right] \tag{4.103}$$

The value of $\sigma_M^2$ was also found for the case where each of the M orthogonal signals is known except for phase, and the phase angle has a uniform distribution.[2] For this case

$$\sigma_M^2 = \frac{1}{M}\left[I_0\left(\frac{2E}{N_0}\right) - 1\right] \quad , \text{ and hence} \tag{4.117}$$

$$d = \ell n\left[1 - \frac{1}{M} + \frac{1}{M}\ I_0\left(\frac{2E}{N_0}\right)\right] \quad . \tag{4.118}$$

---

[1] See Section 4.6.     [2] See Section 4.7.

These two cases are the basis for the best approximation available to the problem of a signal of unknown time origin or a signal of unknown frequency or both. For example, we have been unable to find the distribution of likelihood ratio for the case of a signal which is a pulse of unknown carrier phase if the starting time is random and distributed uniformly over a time interval. However, if the problem is changed slightly, so that the starting time is restricted to times spaced approximately a pulse width apart, then pulses starting at different times would be approximately orthogonal, and the case of the signal one of M orthogonal signals known except for phase could be applied. Eq (4.118) should be used with M equal to the ratio of observation time to pulse width. A similar argument applies to the case in which a signal is a pulse known except for phase and center frequency. Eq (4.118) should be used with M taken as the ratio of total bandwidth to signal bandwidth. It should be pointed out that it is not the same to assume that the signal can appear in only a finite number of different positions, even though the positions are close to each other, as to say that the signal can appear anywhere in an interval. There is more uncertainty in the latter case, and the signal cannot be detected as easily.

5.1.5 The Broad Band Receiver and the Ideal Receiver. One common method of detecting pulse signals in a frequency band is to build a receiver whose bandwidth is the entire frequency band. The receiver operating characteristic for such a receiver with a pulse signal of known starting time is calculated in Section 4.4. This is not a truly ideal receiver, and it would be interesting to compare it with an ideal receiver. This can be done using the approximation of the preceding paragraph for the ideal receiver. Since the bandwidth of a pulse is approximately the reciprocal of the pulse width, the parameter M of Section 4.4 and the parameter M in Eq (4.118) are both equal to

the ratio of total bandwidth to pulse bandwidth. Curves showing $\frac{2E}{N_0}$ as a function of d are given in Fig. 5.3 for the approximate ideal receiver and the broad band receiver for several values of M. The expression used for d is Eq (4.71) which holds for large values of M.

5.1.6 Uncertainty and Signal Detectability. In the two cases discussed in Section 5.1.4, where the signal considered is one of M orthogonal signals, the uncertainty of the signal is a function of M. This gives us an opportunity to study the effect in those two cases of uncertainty on signal detectability. In the approximate evaluation of the receiver built to detect the presence of a signal when the signal is one of M orthogonal functions, the curves of Fig. 5.1 are used with the detection index d given by

$$ d = \ln\left[1 - \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2E}{N_0}\right)\right] \tag{4.103} $$

This equation can be solved for the signal energy.

$$ \frac{2E}{N_0} = \ln\left[1 - M + Me^d\right] $$

$$ \approx \ln M + \ln(e^d - 1) \quad , \tag{5.2} $$

the approximation holding for large $\frac{2E}{N_0}$.[1] From this equation it can be seen that the signal energy is approximately a linear function of $\ln M$ when the detection index d, and hence the ability to detect signals, is kept constant.[2]

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[1] If $\frac{2E}{N_0} > 3$, the error is less than 10%.

[2] It might be suspected that $\frac{2E}{N_0}$ is a linear function of the entropy $= -\sum p_i \ln p_i$, where $p_i$ is the probability of the $i^{th}$ orthogonal signal. This is not the case, except when all the $p_i$ are equal. The expression which occurs in this more general case is:

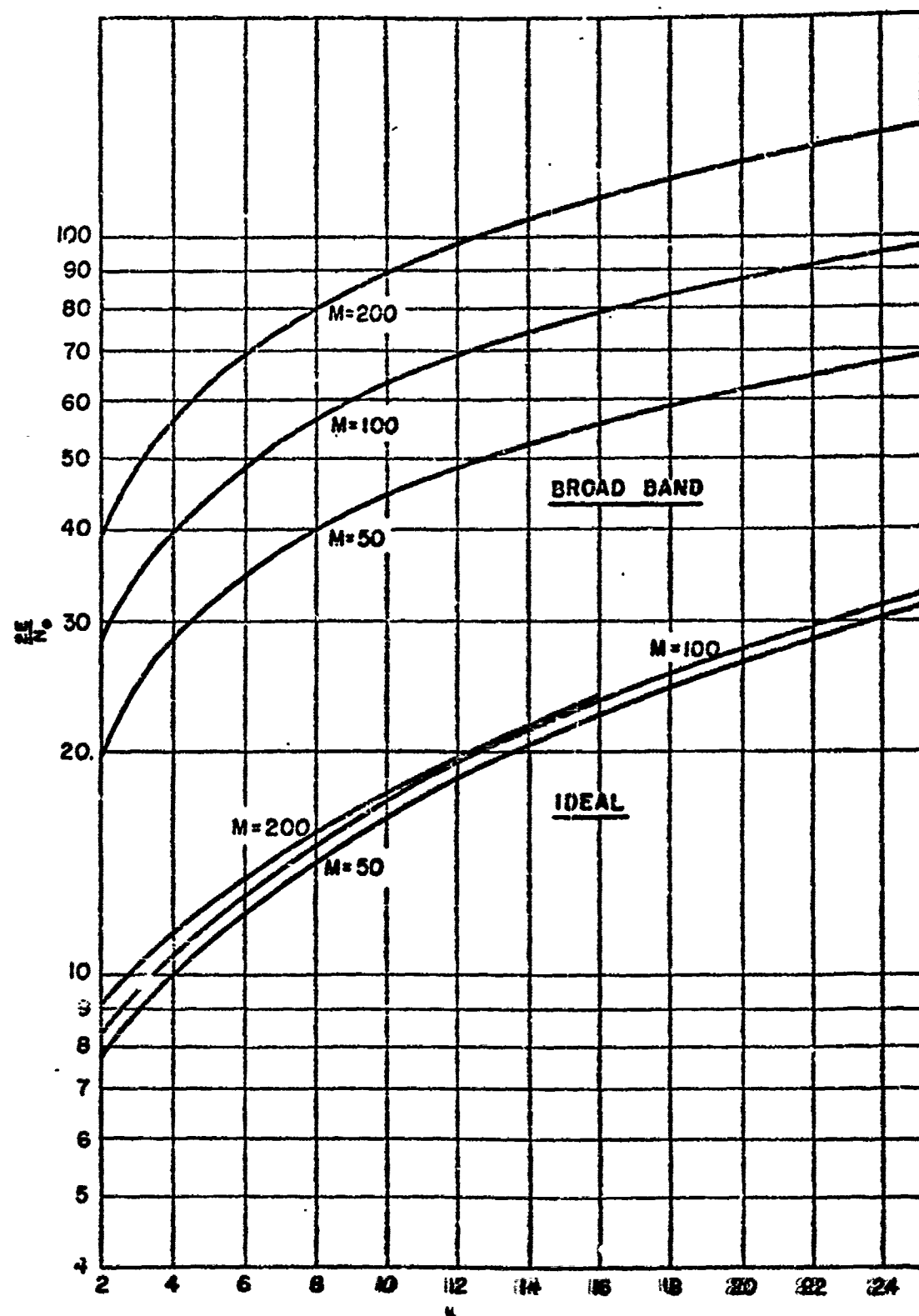$$ \frac{2E}{N_0} \approx -\ln\left[\sum p_i^2\right] + \ln(e^d - 1) $$

FIG. 5.3. COMPARISON OF IDEAL AND BROAD BAND
RECEIVERS.

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5.2 Receiver Design. There are a few cases when the receiver design is simple to specify if the noise is Gaussian. If, for example, not only the noise, but also the signal are Gaussian, and both have a uniform spectrum over their bandwidth, then the optimum receiver simply measures the energy which comes in during the observation period. The simple relation between energy and likelihood ratio is given by Eq (4.41) of Section 4.4.

The simplest remaining case is that in which the signal is known exactly. Then the theory specifies that the receiver find the cross correlation between the expected signal and the receiver input, i.e.,

$$\int_0^T s(t) \, x(t) \, dt, \tag{5.3}$$

where $s(t)$ is the expected signal and $x(t)$ is the receiver input, and the observation interval is from $t = 0$ to $t = T$. The ratio of this cross correlation to the noise power per unit bandwidth is one-half the natural logarithm of the likelihood ratio.[1] Several elaborate correlating devices have been built recently.[2]

There is, in this case, a simple means of obtaining the correlation, if the signal is simple in form, for example, a pulse. If a filter can be designed with impulse response

$$h(t) = s(T-t) \qquad \text{if } 0 \le t \le T \ ,$$
$$= 0 \qquad \text{otherwise}, \tag{4.10}$$

and the receiver input applied to the filter, then the output at time T will be

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[1] See page 9, Eq (4.1b).

[2] Harrington and Rogers, Ref. 16; Nuttilng and Meade, Ref. 17; Lee Cheatham, and Wiesner, Ref. 18; Levin and Reintzes, Ref. 19.

$$\int_{-\infty}^{T} x(\tau) \, h(T-\tau) \, d\tau = \int_{0}^{T} x(\tau) \, s(\tau) \, d\tau \quad , \qquad (4.11)$$

which is the required correlation. It turns out that this is the same filter specified by Middleton, Van Vleck, Wiener, North, and Hansen as the filter which maximizes signal-to-noise ratio.[1]

If the signal being sought is an amplitude modulated signal known except for carrier phase, than the ideal receiver has a filter like the one specified in the previous paragraph designed for any particular phase. The receiver input is applied to this filter, and the output is an rf (or more likely, if) voltage. It turns out that the envelope of this voltage is the required quantity. Its relation to likelihood ratio is derived in Section 4.3 and presented in Eqs (4.19) and (4.29).

A look at the general equation for likelihood ratio

$$\ell(x) = \int_{R} \exp\left[-\frac{T}{?}_{N_0}\right] \exp\left[\frac{2}{N_0} \int_{0}^{T} x(t) \, s(t) \, dt\right] dP_S(s) \quad . \qquad (3.7b)$$

suggests the following method for designing the optimum receiver for signal detection. First find the correlation as described above, between the receiver input and each possible expected signal. Next, divide each by $N_0$, the noise per unit bandwidth, and find the exponential function of each. Finally, find the weighted average of all those quantities. The hard part is to find the cross correlation between each expected signal and the receiver input. This means that the ideal filter and associated amplifiers are needed for each expected signal, or essentially a separate receiver for each expected signal. In most

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[1] Lawson and Uhlenbeck, Ref. 1, p. 206; North, Ref. 11.

cases this is out of the question. In the cases studied in Sections 4.2, 4.3, 4.4, and 4.6, some peculiarity of the set of expected signals made a simpler ideal receiver possible.

There is another noteworthy case. If the signal is known except for starting time, then it is sufficient to look at the same ideal filter at different times rather than to have a different filter for each starting time.

For even a simple square pulse, it is impossible to synthesize the ideal filter exactly. Just how critical, then, is the design of the ideal filter? This can be answered by finding how well signal detection can be accomplished with an approximation to the ideal filter.

For simplicity, consider the case of the signal known exactly. The results for this will follow with little modification for the other cases where the ideal filter is used. The theory specifies that the response of a certain filter to the receiver input be observed at a certain instant. Once it is known that the ideal receiver has this form, it is clear that this filter must be the one which maximizes the instantaneous signal output voltage (or power), the noise rms voltage (or average power) being kept fixed. This is the reason the filter which other authors have found maximizes signal-to-noise ratio is the one which is the absolute optimum for this case.[1]

If a filter can be built for which the output ratio of peak signal to rms noise is nearly the same as that obtained with an ideal filter, then this filter will give results nearly as good as the ideal filter. The noise power at the output of a filter with transfer function $H(\omega)$ is equal to

$$N = N_0 \int_{-\infty}^{\infty} H(\omega) \overline{H(\omega)} \, d\omega \qquad (5.4)$$

---

[1] See footnote, p. 65.

where $N_o$ is the noise power per unit bandwidth of the input noise. By Parseval's theorem,[1] and the fact that $h(t)$, the impulse response, is the Fourier transform of $H(\omega)$.

$$N = N_o \int_{-\infty}^{\infty} H(\omega)\, \overline{H(\omega)}\, d\omega$$

$$= N_o \int_{-\infty}^{\infty} h(t)\, \overline{h(t)}\, dt. \tag{5.5}$$

In the case of the ideal filter, Eq. 4.10 can be applied, and the result is

$$N = N_o \int_0^T s(T-\tau)^2\, d\tau = N_o E \tag{5.6}$$

where $E$ is the signal energy. The peak voltage output if there is signal but no noise is

$$\int_0^T s(t)^2\, dt = E \quad , \tag{5.7}$$

and hence the peak signal power at the output is $E^2$. The ratio of peak signal power to average noise power is thus $\dfrac{E}{N_o}$ for the ideal case.

For the particular case of the signal consisting of a single rectangular pulse, if an RC filter is used with time constant 80% of the pulse duration, the receiver operating characteristic will be the same as if the ideal filter were used and the signal reduced 0.90 db. This is derived in Appendix F. Several other pulse cases have been treated and the results for the best filter of each type are summarized in the following table:

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[1] Titchmarsh, *An Introduction to the Theory of Fourier Integrals*, Oxford University Press, 1937, p. 50.

TABLE II

| Pulse | Filter | Equivalent Loss in Signal Strength |
|---|---|---|
| Gaussian | Rectangular Passband ⎫ | 0.98 db[1] |
| Rectangular | Gaussian Passband ⎭ | |
| Rectangular | Rectangular Passband | 0.83 db[1] |
| Rectangular | Simple RC Filter (or Single Tuned Circuit) | 0.90 db |
| Rectangular | (impulse response) | 0.51 db |
| (rising ramp pulse) | (impulse response) | 1.62 db |
| (Exponential Decay) | Simple RC Filter (or Single Tuned Circuit) | 2.67 db |

The minimum equivalent loss was obtained by adjusting the bandwidth of the filter. Thus in detecting pulses the form of the filter passband is relatively unimportant. However, it is important to have the correct filter bandwidth. This is essentially the present-day attitude in building receivers for receiving pulses of known frequency.

## 5.3 Conclusions

Part II of The Theory of Signal Detectability consists of the application of the theory presented in Part I to some special cases of signal detection problems in order to obtain information on (1) the design of optimum receivers for the detection of signals, and (2) the performance of these receivers.

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[1] These cases are derived in Lawson and Uhlenbeck, Ref. 1, p. 206.

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The special cases which are presented were chosen from the simplest problems in signal detection which closely represent practical situations. They are listed in Table I along with examples of engineering problems in which they find application.

## TABLE I

| Section | Description of Signal Ensemble | Application |
|---------|-------------------------------|-------------|
| 4.2 | Signal Known Exactly | Coherent radar with a target of known range and character |
| 4.3 | Signal Known Except for Phase | Ordinary pulse radar with no integration and with a target of known range and character. |
| 4.4 | Signal a Sample of White Gaussian Noise | Detection of noise-like signals; detection of speech sounds in Gaussian noise. |
| 4.5 | Video Design of a Broad Band Receiver | Detecting a pulse of known starting time (such as a pulse from a radar beacon) with a crystal-video or other type broad band receiver. |
| 4.6 | A Radar Case (A train of pulses with incoherent phase) | Ordinary pulse radar with integration and with a target of known range and character. |
| 4.8 | Signal One of M Orthogonal Signals | Coherent radar where the target is at one of a finite number of non-overlapping positions. |
| 4.9 | Signal One of M Orthogonal Signals Known Except for Phase | Ordinary pulse radar with no integration and with a target which may appear at one of a finite number of non-overlapping positions. |

In the last two cases the uncertainty in the signal can be varied, and some light is thrown on the relationship between uncertainty and the ability to detect signals. The variety of examples presented should serve to suggest methods for attacking other simple signal detection problems and to give insight into problems too complicated to allow a direct solution.

It should be borne in mind that this report discusses the detection of signals in noise; the problem of obtaining information from signals or about signals, except as to whether or not they are present, is not discussed. Furthermore, in treating the special cases, the noise was assumed to be Gaussian.[1]

In addition to general remarks on receiver design,[2] most sections on special cases include specific information describing the simplest design for the optimum receiver for the case considered in those sections.

For the simple cases, the design indicated corresponds closely to the design indicated by the type of analysis in which signal to noise ratio is maximized. For the more complicated cases, the design suggested is usually impractical. For some problems it may never be practical to attempt to build an optimum system. For others, however, engineers equipped with a good understanding of statistical methods and their application to the problem of signal detectability, and to communication theory in general, will undoubtedly invent systems which approach the optimum system.

For each special case treated in this report, at least an approximation is given for the receiver performance. Receiver performance received primary emphasis because it has generally been slighted in previous work. It is

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[1] See the footnote on page 4 with reference to the spectrum of the assumed noise.

[2] See Section 5.2.

important to know the performance which could be obtained from an optimum receiver even if an optimum receiver cannot be built, since this gives an upper bound on the performance which can be obtained with any receiver in a given situation, and since this also gives an upper bound on what can possibly be accomplished by improvements in receiver design.

### APPENDIX D

#### The Sampling Theorem

Suppose $f(t)$ is a measureable function which is defined for $0 \leq t \leq T$. Then $f(t)$ can be expanded in a Fourier series in this interval. The frequency of any term in the series is an integral multiple of $1/T$. Suppose there are no terms in the series with frequency above $W$.[1] This makes the function band limited. Denote by $\psi_m(t)$ the function

$$\psi_m(t) = \frac{\sin\left[\pi(2WT)\left(\frac{t}{T} - \frac{m}{2WT}\right)\right]}{(2WT)\sin\left[\pi\left(\frac{t}{T} - \frac{m}{2WT}\right)\right]} \tag{D.1}$$

Then

$$f(t) = \sum_{m=1}^{2WT} f\left(\frac{m}{2WT}\right)\psi_m(t) \tag{D.2}$$

Furthermore, the functions $\psi_m$ are orthogonal on the interval $0 \leq t \leq T$,

$$\int_0^T \psi_m(t)\,\psi_k(t)\,dt = \frac{\delta_{km}}{2W} \tag{D.3}$$

and

$$\int_0^T \psi_m(t)\,dt = \frac{1}{2W} , \tag{D.4}$$

where $\delta_{km}$ is the Kronecker delta function, which is zero if $k \neq m$ and unity if $k = m$.

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[1] We shall assume $2WT$ is an odd integer. This equivalent to choosing the limit of the band half way between the frequency of the last non zero term in the Fourier series and the frequency of the next term (which, of course, has a zero coefficient).

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It follows from Eq (D.2) and Eq (D.3) that

$$\int_0^T \left[ f(t) \right]^2 \, dt = \frac{1}{2W} \sum_{M=1}^{2WT} \left[ f\left(\frac{m}{2W}\right) \right]^2 \tag{D.5}$$

and from Eqs (D.2) and Eq (D.4)

$$\int_0^T f(t) \, dt = \frac{1}{2W} \sum_{m=1}^{2WT} f\left(\frac{m}{2W}\right) \tag{D.6}$$

Thus the $\psi_m$ functions $\psi_m$ have the same properties for the finite interval which Shannon's interpolation functions have on the infinite interval.[1] It is interesting to note that when 2WT is large, these functions, except the ones near the ends of the interval, are approximately the same as Shannon's.

The Fourier series for $\psi_m(t)$ has no terms with frequency above W. It is, in exponential form,

$$\psi_m(t) = \frac{1}{2WT} \sum_{n=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \exp\left[ j \, \frac{-2\pi \, mn}{2WT} \right] \exp\left[ j \, \frac{2\pi \, nt}{T} \right] \tag{D.7}$$

This can be shown by expressing the sine functions in Eq (D.1) as exponentials and using the algebraic identity

$$\frac{a^{n+1} - a^{-n-1}}{a - a^{-1}} = \sum_{k=-n}^{n} a^k \tag{D.8}$$

Formula (D.4) can be proved by integrating Eq (D.7) directly. Note that the only term which contributes to the integral is the term for which $n = 0$.

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[1] See Shannon, Ref. 21.

Formula (D.3) can best be proved also by using the Fourier series.

$$\int_0^T \psi_m(t)\, \overline{\psi_k(t)}\, dt$$

$$= \frac{1}{(2WT)^2} \int_0^T \sum_{n=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \exp\left[j\, \frac{-2\pi mn}{2WT}\right] \exp\left[j\, \frac{2\pi nt}{T}\right] \sum_{p=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \overline{\exp\left[j\, \frac{-2\pi kp}{2WT}\right] \exp\left[j\, \frac{2\pi pt}{T}\right]}\, dt$$

$$= \frac{1}{(2WT)^2} \sum_{n=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \sum_{p=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \exp\left[j\, \frac{-2\pi(mn-kp)}{2WT+1}\right] \int_0^T \exp\left[j\, \frac{2\pi t(n-p)}{T}\right] dt$$

$$= \frac{1}{(2WT)^2} \sum_{n=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \sum_{p=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \exp\left[j\, \frac{-2\pi(mn-kp)}{2WT}\right] T\delta_{np}$$

$$= \frac{T}{(2WT)^2} \sum_{n=-\left(WT-\frac{1}{2}\right)}^{WT-\frac{1}{2}} \exp\left[j\, \frac{-2\pi n(m-k)}{2WT}\right] .$$

If $m = k$, each of the $2WT$ terms in the sum is unity. If $m \neq k$, the terms in the sum are equally spaced around the unit circle in the complex plane and must sum to zero. Thus

$$\int_0^T \psi_m(t)\, \psi_k(t)\, dt = \frac{\delta_{km}}{2W} ,$$

which was to be proved.

The validity of the expansion in equation (D.2) follows from the fact that the functions $\psi_m(t)$ are 2WT linearly independent linear combinations of the 2WT functions

$$\exp\left[j\ \frac{2\pi n t}{T}\right]\ \frac{1}{2} - WT \leq n \leq WT - \frac{1}{2}$$

which are used in the Fourier series expansion. Thus any function which can be expanded in a Fourier series with only the first 2WT terms can also be expanded in a unique way in terms of the functions $\psi_m$.

There is an alternate form of the sampling theorem for band limited signals. With this form the signal function can be described by giving sample values of the envelope and phase of the signal, and hence this form is often convenient to use in describing rf signals.

Suppose the function $f(t)$, when expanded in a Fourier series on the interval $0 \leq t \leq T$ has only a finite number of terms in its expansion, and suppose they are included in the terms ranging from frequency $f_1$ to frequency $f_2$.[1] The bandwidth then could be defined as

$$W = f_2 - f_1 + \frac{1}{T} , \tag{D.9}$$

and the center frequency is

$$\frac{\omega}{2\pi} = \frac{f_2 + f_1}{2} \tag{D.10}$$

Then the Fourier series can be written

$$f(t) = \sum_{-m}^{m} a_k \cos\left[\left(\omega + \frac{2\pi}{T} k\right) t\right] + b_k \sin\left[\left(\omega + \frac{2\pi}{T}k\right)t\right] \tag{D.11}$$

$$f(t) = R\left\{\sum_{-m}^{m} \left(a_k - ib_k\right)\exp\left[i\left(\omega + \frac{2\pi k}{T}\right) t\right]\right\} \tag{D.12}$$

where R means "the real part of", and $m = \frac{1}{2}(WT - 1)$.

___

[1] We shall assume $Tf_2 - Tf_1$ is an even integer and that $Tf_1 \geq 1$.

$$f(t) = R\left\{ \exp\left[i\omega t\right]\sum_{-m}^{m}(a_k - ib_k)\exp\left[i\frac{2\pi kt}{T}\right]\right\}$$

$$f(t) = R\left\{ \exp\left[i\omega t\right]\left(\sum_{-m}^{m} a_k \exp\left[i\frac{2\pi kt}{T}\right] + i\sum_{-m}^{m} b_k \exp\left[i\frac{2\pi kt}{T}\right]\right)\right\}$$

$$= R\left\{ \exp\left[i\omega t\right]\left(x(t) - iy(t)\right)\right\}$$

$$= x(t)\cos\omega t + y(t)\sin\omega t \qquad (D.13)$$

where

$$x(t) = \sum_{-m}^{m}\frac{a_k}{2}\exp\left[i\frac{2\pi kt}{T}\right], \text{ and}$$

$$\qquad\qquad (D.14)$$

$$y(t) = -\sum_{-m}^{m}\frac{b_k}{2}\exp\left[i\frac{2\pi kt}{T}\right].$$

The functions $x(t)$ and $y(t)$ meet the conditions of the first form of the sampling theorem, for a signal with frequencies no higher than $\frac{W}{2}$. They can be expressed therefore in the form

$$x(t) = \sum_{k=1}^{WT} x\left(\frac{k}{W}\right)\psi_k(t)$$

$$y(t) = \sum_{k=1}^{WT} y\left(\frac{k}{W}\right)\psi_k(t) \qquad (D.15)$$

Where the $\psi$ functions are defined for a signal with no frequencies above $\frac{W}{2}$. Thus the original function can be written as

$$f(t) = \sum_{k=1}^{WT} x\left(\frac{k}{W}\right)\psi_k(t)\cos\omega t + \sum_{k=1}^{WT} y\left(\frac{k}{W}\right)\psi_k(t)\sin\omega t,$$

$$\qquad\qquad (D.16)$$

and the function $f(x)$ can be represented by giving the sample values $x\left(\frac{k}{W}\right)$ and $y\left(\frac{k}{W}\right)$.

Since $f(t)$ can be expressed in the form

$$f(t) = x(t) \cos \omega t + y(t) \sin \omega t \tag{D.13}$$

and $x(t)$ and $y(t)$ are limited to frequencies less than $\frac{W}{2}$ which is less than $\frac{\omega}{2\pi}$, the envelope of $f(t)$ is

$$r(t) = \sqrt{x(t)^2 + y(t)^2} \tag{D.17}$$

The angle $\Theta(t)$ defined by

$$\cos \Theta(t) = \frac{x(t)}{r(t)}$$

$$\sin \Theta(t) = -\frac{y(t)}{r(t)} \tag{D.18}$$

can be considered as the phase of the signal, since

$$f(t) = r(t) \cos \Theta(t) \cos \omega t - r(t) \sin \Theta(t) \sin \omega t$$

$$= r(t) \cos \left[\omega t + \Theta(t)\right] \tag{D.19}$$

Note that the sample values $x_i$ and $y_i$ can be obtained from sample values of $r$ and $\Theta$,

$$x_i = x\left(\frac{i}{W}\right) = r\left(\frac{i}{W}\right) \cos \left[\Theta\left(\frac{i}{W}\right)\right] = r_i \cos \Theta_i$$

$$y_i = r_i \sin \Theta_i \tag{D.20}$$

Thus the function $f(t)$ may be represented by giving the sample values of its amplitude and phase at points spaced $\frac{1}{W}$ apart through the observation interval.

## APPENDIX E

The integral

$$\exp(-b^2) \int_0^\infty \left[ I_0(b\alpha) \right]^2 \alpha \exp\left[ -\frac{\alpha^2}{2} \right] d\alpha \qquad (E.1)$$

is required.

The integral

$$\int_0^\infty \alpha^{2n+1} I_0(b\alpha) \exp\left[ -\frac{\alpha^2}{2} \right] d\alpha$$

$$= n! 2^n \exp\left[ \frac{b^2}{2} \right] F\left( -n, 1; -\frac{b^2}{2} \right)$$

$$= n! 2^n \exp\left[ \frac{b^2}{2} \right] \sum_{k=0}^{n} \frac{n! b^{2k}}{(n-k)! k! k! 2^k} , \qquad (E.2)$$

where $F\left( -n, 1; -\frac{b^2}{2} \right)$ is the confluent hypergeometric function.[1] The function $I_0(b\alpha)$ can be expanded in a power series

$$I_0(b\alpha) = \sum_{n=0}^{\infty} \frac{b^{2n} \alpha^{2n}}{2^{2n} n! n!} \qquad (E.3)$$

Then the integral (E.1) can be written

$$\exp(-\gamma^2) \int_0^\infty \left[ I_0(b\alpha) \right]^2 \alpha \exp\left[ -\frac{\alpha^2}{2} \right] d\alpha = \qquad (E.4)$$

(Substituting (E.3) for $I_0(b\alpha)$)

$$= \exp(-b^2) \int_0^\infty \sum_{n=0}^{\infty} \frac{b^{2n} \alpha^{2n}}{2^{2n} n! n!} I_0(b\alpha) \alpha \exp\left[ -\frac{\alpha^2}{2} \right] d\alpha$$

$$(E.5)$$

---

[1] Lawson and Uhlenbeck, Ref. 1, p. 174.

$$= \exp(-b^2) \sum_{n=0}^{\infty} \int_0^{\infty} \frac{b^{2n} \alpha^{2n+1}}{2^{2n} n! n!} I_0(b\alpha) \exp\left[-\frac{\alpha^2}{2}\right] d\alpha \qquad (E.6)$$

(Substituting from (E.2))

$$= \exp(-b^2) \sum_{n=0}^{\infty} \frac{b^{2n}}{2^{2n} n! n!} n! 2^n \exp\left[\frac{b^2}{2}\right] \sum_{k=0}^{n} \frac{n! b^{2k}}{(n-k)! k! k! 2^k} \qquad (E.7)$$

$$= \exp\left[-\frac{b^2}{2}\right] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b^{2n+2k}}{2^{n+k}(n-k)! k! k!} \qquad (E.8)$$

(Rearranging the terms in the double sum)

$$= \exp\left[-\frac{b^2}{2}\right] \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{b^{2n+2k}}{2^{n+k}(n-k)! k! k!} \qquad (E.9)$$

$$= \exp\left[-\frac{b^2}{2}\right] \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{b^{4k} b^{2n-2k}}{2^{2k} 2^{n-k} k! k! (n-k)!} \qquad (E.10)$$

(Letting m = n-k)

$$= \exp\left[-\frac{b^2}{2}\right] \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{b^{4k} b^{2m}}{2^{2k} k! k! m! 2^m} \qquad (E.11)$$

$$= \exp\left[-\frac{b^2}{2}\right] \sum_{k=0}^{\infty} \frac{b^{4k}}{2^{2k} k! k!} \sum_{m=0}^{\infty} \frac{b^{2m}}{m! 2^m} \qquad (E.12)$$

$$= \exp\left[-\frac{b^2}{2}\right] I_0(b^2) \exp\left[\frac{b^2}{2}\right] = I_0(b^2) \qquad (E.13)$$

The steps in this derivation which must be justified are interchanging the order of integration and summation at step (E.6) and rearranging the double sum, at steps (E.9) and (E.12). It is easy to show that the integral (E.4) exists. The integrands in (E.6) are uniformly bounded by the integrand in (E.4). Thus

the integrals in (Σ.6) converge uniformly, and the order of integration and summation can be interchanged. As for rearranging double sums, this is possible since all the terms are positive, and hence the convergence is absolute.

## APPENDIX F

Let us consider a simple case of approximating the ideal filter by some other filter. Suppose $s(t)$ is a rectangular pulse of energy $E$ and width $d$. Then

$$s(t) = \sqrt{\frac{E}{d}} \quad \text{if } 0 \leq t \leq d$$
$$= 0 \quad \text{otherwise} \tag{F.1}$$

Suppose the filter is made up of a single resistor and a single condenser, with an amplifier or attenuator, whichever is needed to make the noise power at the output $N_oE$ as in the ideal case. Then the impulse response is of the form

$$h(t) = h_o e^{-\frac{t}{\tau}} \quad \text{if } t \geq 0$$
$$= 0 \quad \text{otherwise} \tag{F.2}$$

where $\tau$ is the time constant of the filter and $h_o$ is a constant depending on the gain of the amplifier or attenuator. The requirement that the noise power at the output be $N_oE$ is, by (5.6), equivalent to requiring that

$$E = \int_{-\infty}^{\infty} \left[ h(t) \right]^2 dt, \tag{F.3}$$

or

$$E = \int_0^{\infty} h_o^2 e^{-\frac{2t}{\tau}} dt = \frac{h_o^2 \tau}{2} \tag{F.4}$$

which yields

$$h_o^2 = \frac{2E}{\tau}, \quad \text{and} \tag{F.5}$$

$$h(t) = \sqrt{\frac{2E}{\tau}} e^{-\frac{t}{\tau}} \quad \text{if } t \geq 0$$
$$= 0 \quad \text{otherwise} \tag{F.6}$$

The response $V(t)$ of this filter to the pulse $s(t)$ is, by (4.11),

$$V(t) = \int_{-\infty}^{t} s(\lambda) h(t-\lambda) d\lambda. \tag{F.7}$$

Substitutions from (F.1) and (F.6) for $s(t)$ and $h(t)$ give

$$V(t) = \int_0^t \sqrt{\frac{E}{d}} \cdot \sqrt{\frac{2E}{\tau}} \, e^{-\frac{(t-\lambda)}{\tau}} \, d\lambda \quad \text{if } 0 \leq t \leq d$$

$$= \int_0^d \sqrt{\frac{E}{d}} \sqrt{\frac{2E}{\tau}} \, e^{-\frac{(t-\lambda)}{\tau}} \, d\lambda \quad \text{if } t > d. \tag{F.8}$$

These integrals can be evaluated easily, and

$$V(t) = E\sqrt{\frac{2\tau}{d}} \left( 1 - e^{-\frac{t}{\tau}} \right) \quad \text{if } 0 \leq t \leq d$$

$$V(t) = E\sqrt{\frac{2\tau}{d}} \left( 1 - e^{-\frac{d}{\tau}} \right) e^{-\frac{(t-d)}{\tau}} \quad \text{if } t > d. \tag{F.9}$$

$V(t)$ increases with time if $t < d$ and decreases with time if $t > d$, so it must have its maximum value at $t = d$. That maximum value is

$$V_{max} = E\sqrt{\frac{2\tau}{d}} \left( 1 - e^{-\frac{d}{\tau}} \right). \tag{F.10}$$

In Fig. F.1, $V_{max}/E$ is plotted as a function of $\frac{\tau}{d}$. It is seen that at $\frac{\tau}{d} = 0.8$ approximately, $V_{max}$ has a maximum, and at this point $V_{max}$ is approximately $0.9E$.

For this particular case, if the RC filter with time constant $\tau = 0.8d$ is used in place of the ideal filter, the reliability of signal detection will be the same as if the ideal filter were used and the signal amplitude were reduced to ninety per cent, or 0.90 decibel.
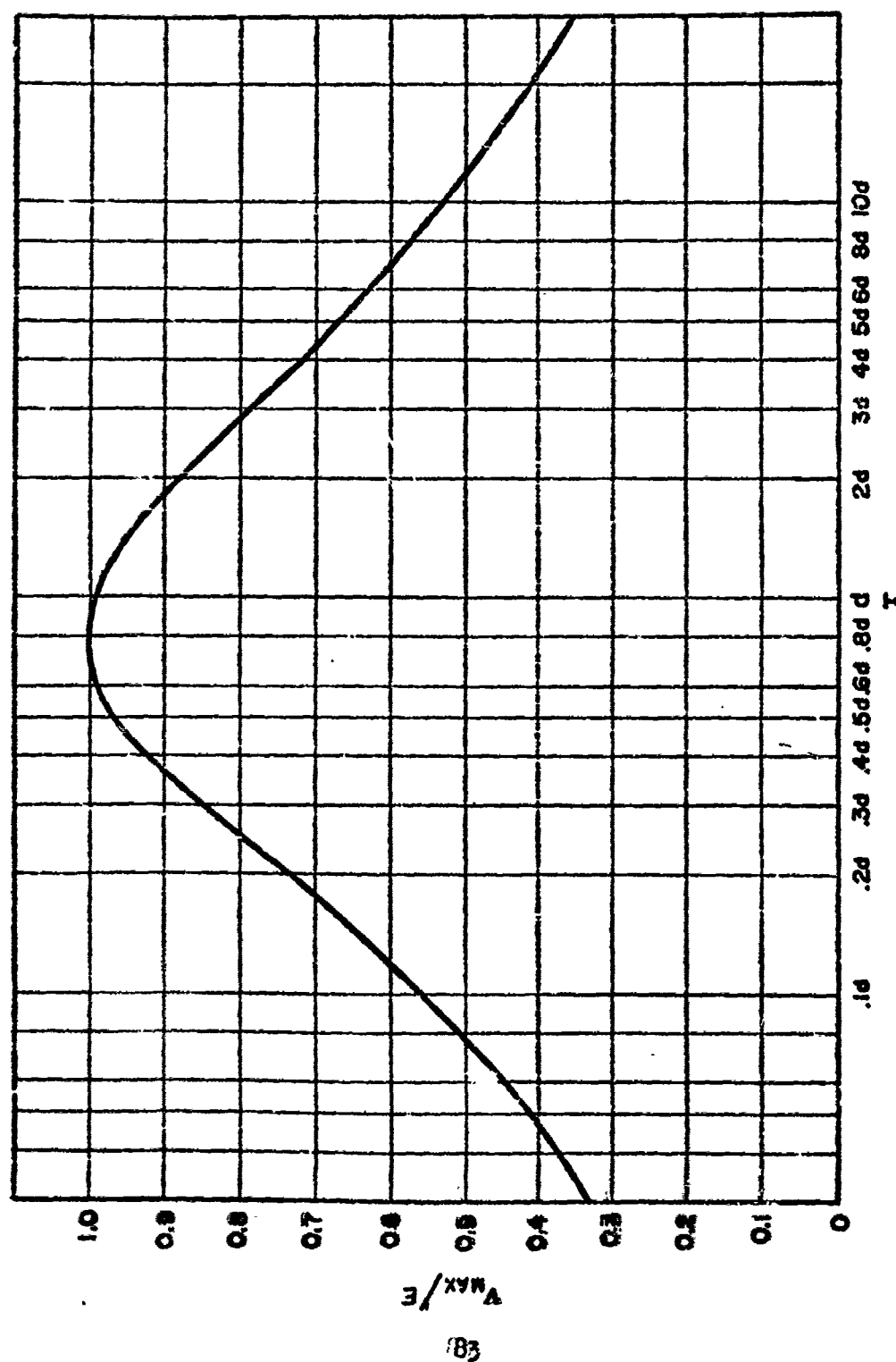
FIG. F.1

MAXIMUM RESPONSE OF RC FILTER TO A
RECTANGULAR PULSE AS A FUNCTION OF
FILTER TIME CONSTANT.

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## LIST OF SYMBOLS

| | |
|---|---|
| A | The event "The operator says there is signal plus noise present," or a criterion, i.e., the set of receiver inputs for which the operator says there is a signal present. |
| $A_1(\beta)$ | Any criterion A which maximizes $P_{SN}(A) - \beta\, P_N(A)$, i.e., an optimum criterion of the first type. |
| $A_2(k)$ | Any criterion A for which $P_N(A) \leq k$, and $P_{SN}(A)$ is maximum, i.e., an optimum criterion of the second type. |
| CA | The event "The operator says there is noise alone." |
| d | A parameter describing the ability of a receiver to detect signals. (See Section 5.1 and Fig. 5.1.) |
| $E$, $E(s)$ | The signal energy. |
| $E^n$ | The n-dimensional Euclidean space. |
| $f_N(x)$ | The probability density for points x in R if there is noise alone. |
| $f_{SN}(x)$ | The probability density for points x in R if there is signal plus noise. |
| $F_N(\beta)$, $F_N(\ell)$ | The complementary distribution function for likelihood ratio if there is noise alone, i.e., $F_N(\beta)$ is the probability that the likelihood ratio will be greater than $\beta$ if there is noise alone. |
| $F_{SN}(\beta)$, $F_{SN}(\ell)$ | The complementary distribution function for likelihood ratio if there is signal plus noise. |
| k | A symbol used primarily for the upper bound placed on false alarm probability $P_N(A)$ in the definition of the second kind of optimum criterion. |
| $\ell(x)$ | The likelihood ratio for the receiver input x. $\ell(x) = \dfrac{f_{SN}(x)}{f_N(x)}$ . |
| n | The dimension of the space of receiver inputs. $n = 2WT$ . |
| N | The event "There is noise alone," or the noise power. |
| $N_o$ | The noise power per unit bandwidth. $N_o = N/W$ . |
| $P_N(A)$ | The probability that the operator will say there is signal plus noise if there is noise alone, i.e., the false alarm probability. |

$P_{SN}(A)$ — The probability that the operator will say there is signal plus noise if there is signal plus noise, i.e., the probability of detection.

$P_x(SN)$ — The a posteriori probability that there is signal plus noise present. (See Sections 1 3 and 2.3.)

$P_S(s)$ — The probability measure defined on R for the set of expected signals.

R — The space of all receiver inputs. (The set of all possible signals is the same space.)

s — A signal $s(t)$, which may also be considered as a point s in R with coordinates $(s_1, s_2, \ldots, s_n)$.

SN — The event "There is signal plus noise."

t — Time.

T — The duration of the observation.

W — The bandwidth of the receiver inputs.

x — A receiver input $x(t)$, which may also be considered as a point x in R with coordinates $(x_1, x_2, \ldots, x_n)$

$\beta$ — A symbol usually used for the likelihood ratio level of an optimum criterion.

$\mu_{SN}(z)$ — The mean of the random variable z if there is signal plus noise.

$\mu_N(z)$ — The mean of the random variable z if there is noise alone.

$\sigma_N^2(z)$ — The variance of the random variable z if there is noise alone.

$\sigma_N^2$ — The variance of likelihood ratio if there is noise alone.

Note: The terms "normal distribution" and "Gaussian distribution" have been used interchangeably in this report.

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